AN INTRODUCTION TO DERIVED ALGEBRAIC GEOMETRY

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0. Overview

These lecture notes aim to provide a working knowledge of the languages of ∞-categories and derived algebraic geometry. Given the amount of material this humble author was tasked to cover, sacrifices in the exposition have had to be made. The reader who is disturbed by the omission of many proofs is begged to consult the lecture notes [Kha3] that cover most of this material in much more detail.

There are several excellent textbook accounts such as [HTT, Ci] focused on developing the extensive technical machinery necessary to justify the existence of the theory of ∞-categories. Here we'll only give a brief and informal introduction, taking well-foundedness of the theory for granted. We'll focus on understanding how this language is useful in the study of

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"derived" or "homotopical" objects that are most naturally regarded up to something weaker than isomorphism (such as quasi-isomorphism, equivalence, or weak equivalence). Relevant examples for us will be:

- the singular (co)chain complex of a topological space,
- the cotangent complex of a scheme or stack,
- the derived category of (quasi-)coherent sheaves on a scheme or stack,
- stacks and higher stacks,
- derived schemes and derived stacks.

Indeed, it is only by working ∞-categorically that we are able to take advantage of the descent properties satisfied by these objects. For example, we have:

Theorem 0.1.

(i) *The assignment $X \mapsto \mathrm{C}^\bullet(X;\mathbf{Z})$, sending a topological space to its complex of singular cochains, is a sheaf with values in the derived ∞-category of abelian groups.*

(ii) *The assignment $X \mapsto \mathrm{D}_{\mathrm{qc}}(X)$, resp. $X \mapsto \mathrm{D}_{\mathrm{coh}}(X)$, sending a scheme to its stable ∞-category of quasi-coherent (resp. coherent) sheaves, is a sheaf of ∞-categories (for the Zariski, étale, and even fpqc topology).*

This ability to speak about sheaves of derived objects is one of the fundamental features of ∞-categories and has many far-reaching consequences (of which we shall only see a small glimpse). Note in contrast that $X \mapsto \mathrm{C}^\bullet(X;\mathbf{Z})$ does not satisfy descent when regarded with values in the usual derived category of abelian groups, nor in the category of chain complexes. Likewise, the descent condition fails also if we replace $X \mapsto \mathrm{D}_{\mathrm{qc}}(X)$ by the assignment sending X to the usual derived category (regarded as an ordinary category).

Another key aspect of ∞-category theory is a very flexible framework for deriving functors. Recall that the usual framework of homological algebra allows us to derive functors between abelian categories. Using the language of ∞-categories we may even derive constructions like the tangent bundle (or cotangent sheaf); the corresponding derived functor, the derived tangent bundle (or cotangent complex), is naturally defined on the ∞-category of derived stacks, which is of course not a derived category in any traditional sense. In fact, the ∞-categorical approach is advantageous even for deriving abelian categories like the category of quasi-coherent sheaves on a scheme or stack: descent allows us to freely extend constructions defined on affine schemes, thereby bypassing most of the technicalities in classical approaches.

In the second part we will introduce derived stacks and cotangent complexes. Our goal here will be to build up to the following fundamental result (which unfortunately does not seem to be covered in the standard textbooks [Lur, TV2, GR]):

Theorem 0.2. *Let X be a smooth proper scheme over a field k. Let $\mathcal{M}$ be the derived moduli stack $\mathcal{M}_{\mathrm{Vect}(X)}$ of vector bundles on X, $\mathcal{M}_{\mathrm{Coh}(X)}$ of coherent* *sheaves on X, or $\mathcal{M}_{\mathrm{Bun}_G(X)}$ of principal G-bundles on X (for an algebraic group G). Then $\mathcal{M}$ is a derived algebraic stack which is "homotopically smooth" in the sense that its cotangent complex $\mathbf{L}_{\mathcal{M}}$ is perfect. In addition, if X is of dimension $\leqslant d$, then $\mathbf{L}_{\mathcal{M}}$ is of Tor-amplitude $\leqslant d-1$. In particular, it is smooth if X is a curve and quasi-smooth if X is a surface.*

Let $\mathcal{M}_{\mathrm{cl}}$ denote the classical moduli stack of vector bundles (resp. coherent sheaves, principal G-bundles) on X. There is a surjective closed immersion $\mathcal{M}_{\mathrm{cl}} \hookrightarrow \mathcal{M}$, which is an isomorphism if and only if X is a curve, hence if and only if $\mathcal{M}$ is smooth. As soon as X is of dimension 2 or greater, $\mathcal{M}_{\mathrm{cl}}$ is singular with unbounded cotangent complex while $\mathcal{M}$ is still homotopically smooth. In other words, the homotopical smoothness of $\mathcal{M}$ is a property that can be witnessed only through the lens of derived algebraic geometry. This phenomenon, labelled "hidden smoothness" by Kontsevich [Kon], is the source of "virtual" phenomena on $\mathcal{M}$.

We will conclude the notes with a brief introduction to the cohomological approach to (virtual) intersection theory on stacks following [Kha1]. We will see how ∞-categorical descent allows us to work effectively with cohomology and Borel–Moore homology of stacks. Revisiting the theory of virtual fundamental classes using this framework, we will see that it also enables very conceptual proofs of the virtual torus localization formula, with no need for the usual auxiliary technical hypotheses on global smooth embeddings or global resolutions and such.

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1. Derived ∞-categories

1.1. **∞-Categories.** We begin with a "working mathematician's guide" to ∞-categories; see [HTT, Ci] for the rigorous definitions.

Although our main interest is in ∞-categories that arise from algebraic geometry, it will be useful to begin with the study of the "universal" homotopy theory, namely that of ∞-groupoids or homotopy types. We will see that ∞-groupoids play the role of sets in ∞-category theory.

Definition 1.1. A 1-*groupoid* is a 1-category in which all 1-morphisms are invertible.

Example 1.2. Let X be a topological space. Its *fundamental* 1-*groupoid* $\Pi_1(X)$ is defined as follows. Its objects are points of X. Morphisms $x \to y$, where $x, y \in X$, are paths $\gamma : [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$, modulo (endpoint-preserving) homotopy. Composition is defined by concatenation of paths (which is associative up to homotopy).

Definition 1.3. Let us say a continuous map $f : X \to Y$ is a *homotopy* 1-*equivalence* if it induces isomorphisms

$$f_* : \pi_0(X) \to \pi_0(Y) \quad \text{and } f_* : \pi_1(X, x) \to \pi_1(Y, f(x)) \text{ for all } x \in X.$$

The *homotopy category of* 1-*types* is the localization of the category of topological spaces at the class of homotopy 1-equivalences: that is, it is the category formed by formally adjoining inverses to all homotopy 1-equivalences (see [GZ]).

The following classical result essentially follows from the work of Eilenberg–MacLane [EM]:

Proposition 1.4.

(i) *The* 1*-groupoid* $\Pi_1(X)$ *only depends on the* homotopy 1-type *of* X*.*

(ii) *The assignment* $X \mapsto \Pi_1(X)$ *determines an equivalence from the homotopy category of* 1*-types to the homotopy category of groupoids (= the localization of the category of groupoids at equivalences).*

The definition of n-groupoids for higher n is much more subtle, but as a guiding principle we might similarly expect to be able to associate with any topological space X a *fundamental* n*-groupoid* $\Pi_n(X)$ that sees the *homotopy* n*-type* of X, and that this determines an equivalence from the homotopy category of n-types to the homotopy category of n-groupoids (cf. Grothendieck [Gro]).

"Definition" 1.5. Let X be a topological space. An n*-groupoid* (which we think of as $\Pi_n(X)$ for some space X) has:

(i) Objects (corresponding to points of X).

(ii) 1-morphisms (corresponding to paths in X).

(iii) 2-morphisms (corresponding to homotopies between paths in X).

(iv) $\cdots$

A subtlety here is that the composition (concatenation of paths) should only be associative up to (coherent) homotopy. It is an essentially impossible task to write down by hand a full description of all the homotopy coherence data present in the fundamental n-groupoid as soon as n exceeds 3 or 4. To be able to make Definition 1.5 precise we would need some suitable bookkeeping device. Somewhat counterintuitively, it turns out that the theory simplifies considerably once we take the limit $n \to \infty$.

Definition 1.6. Let $\mathbf{\Delta}$ be the category whose objects are the finite sets $[n] := \{0, 1, \ldots, n\}$ for $n \geqslant 0$ and morphisms are order-preserving maps. A *simplicial set* is a functor $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}$. Equivalently, this is a collection of sets $(X_n)_{n \geqslant 0}$ together with "face" and "degeneracy" maps

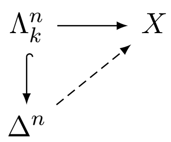
$$d_n^i : X_n \to X_{n-1}, \quad s_n^i : X_n \to X_{n+1}$$

which we often depict by the diagram

$$\cdots \overset{\rightarrow}{\underset{\rightarrow}{\rightrightarrows}} X_2 \overset{\rightarrow}{\rightrightarrows} X_1 \rightrightarrows X_0,$$

only drawing the face maps for simplicity.

Definition 1.7. A *Kan complex* is a simplicial set $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}$ satisfying the Kan condition: for every solid arrow diagram as follows, there exists a lift in the following diagram.



Here Δ^n is the standard n-simplex and Λ^n_k is the result of removing the kth face from the boundary $\partial\Delta^n$. For example, for $n = 2$ such lifts correspond to the existence of composites of any two morphisms ($k = 1$) and left and right inverses to all morphisms ($k = 0$ and $k = 2$).

Theorem 1.8 (Milnor)**.** *Given a topological space X, its fundamental groupoid $\Pi_\infty(X)$ is the simplicial set whose n-simplices are continuous map $\Delta^n_{\mathrm{top}} \to X$ from the topological standard n-simplex. Then $\Pi_\infty(X)$ is a Kan complex, and the assignment $X \mapsto \Pi_\infty(X)$ determines an equivalence from the homotopy category of topological spaces to the homotopy category of Kan complexes.*

Exercise 1.9. Let $\mathcal{C}$ be a category. The *nerve* of $\mathcal{C}$ is the simplicial set $N(\mathcal{C})$ whose n-simplices are n-folds composites of morphisms in $\mathcal{C}$, i.e., diagrams

$$c_0 \to c_1 \to \cdots \to c_n$$

in $\mathcal{C}$. Show that $N(\mathcal{C})$ is a Kan complex if and only if $\mathcal{C}$ is a groupoid.

In view of Theorem 1.8, we will take Kan complexes as a model for ∞-*groupoids*. We can then make the following informal definition:

"Definition" 1.10. An ∞-*category* is a category that is "homotopically enriched" in ∞-groupoids: for any two objects x, y there is a mapping ∞-groupoid

$$\mathrm{Maps}(x, y) \in \mathrm{Grpd}_\infty,$$

and for x, y, z there is a composition map

$$\mathrm{Maps}(x, y) \times \mathrm{Maps}(y, z) \to \mathrm{Maps}(x, z)$$

which is associative up to coherent homotopy.

Remarkably, it turns out that all the relevant homotopy coherence data can be again encoded by the category $\mathbf{\Delta}$. In fact, it is possible to give a precise version of Definition 1.10, where an ∞-category $\mathcal{C}$ is a simplicial diagram $X_\bullet$ of Kan complexes satisfying certain conditions that ensure that it looks like an ∞-categorical version of the nerve, i.e., X_n is the Kan complex of n-fold composites of morphisms in $\mathcal{C}$. We will not make this precise here.

Definition 1.11 (Limits and colimits)**.** Let F be a diagram in an ∞-category $\mathcal{C}$ indexed by an ∞-category I, i.e., a functor of ∞-categories $F : I \to \mathcal{C}$. Suppose given an object $X \in \mathcal{C}$ and a natural transformation $\alpha : X_{\mathrm{cst}} \to F$ where X_{cst} denotes the constant diagram $(i \in I) \mapsto (X \in \mathcal{C})$. We say that the pair (X, α) in $\mathcal{C}$ *exhibits* X *as the limit of* F if for every object $Y \in \mathcal{C}$ the induced functor of mapping ∞-groupoids

$$\mathrm{Maps}_{\mathcal{C}}(Y, X) \to \mathrm{Maps}_{\mathrm{Fun}(I,\mathcal{C})}(Y_{\mathrm{cst}}, F),$$

sending $(Y \to X) \mapsto (Y_{\mathrm{cst}} \to X_{\mathrm{cst}} \to F)$, is invertible. In that case we write

$$X \simeq \varprojlim_{i \in I} F_i.$$

Dually, we may speak of the *colimit* of F, which is the same as the limit of $F^{\mathrm{op}} : I^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$.

Warning 1.12. In the ∞-category of ∞-groupoids, co/limits correspond to *homotopy* co/limits of homotopy types. Thus for example, if X is a topological space with homotopy type $X^{\mathrm{ho}} \in \mathrm{Grpd}_\infty$ and $x \in X$ is a point, the limit in Grpd_∞ of the diagram

$$\mathrm{pt} \xrightarrow{x} X^{\mathrm{ho}} \xleftarrow{x} \mathrm{pt}$$

is the loop space $\Omega_x(X)$, since a commutative square in Grpd_∞ of the form

$$\begin{array}{ccc} Y & \longrightarrow & \mathrm{pt} \\ \downarrow & & \downarrow x \\ \mathrm{pt} & \xrightarrow{x} & X^{\mathrm{ho}} \end{array}$$

encodes a self-homotopy of the constant map $x : Y \to X^{\mathrm{ho}}$. Contrast this with the fact that the limit of the diagram

$$\mathrm{pt} \xrightarrow{x} X \xleftarrow{x} \mathrm{pt}$$

in the category of topological spaces is just pt.

1.2. **Animation.**

Definition 1.13. A category $\mathcal{C}$ is *algebraic* if it is equivalent to

$$\mathrm{Fun}_\Pi(\mathcal{F}^{\mathrm{op}}, \mathrm{Set})$$

for some category $\mathcal{F}$ admitting finite coproducts, where the subscript indicates functors sending finite coproducts in $\mathcal{F}$ to finite products.

Remark 1.14. If $\mathcal{F}$ is generated under finite coproducts by an object $1 \in \mathcal{F}$, so that every object of $\mathcal{F}$ is $1^{\oplus n}$ for some $n \geqslant 0$, then we may think of $\mathrm{Fun}_\Pi(\mathcal{F}^{\mathrm{op}}, \mathrm{Set})$ as the category of sets X equipped with some algebraic operations

$$X^{\times m} \to X^{\times n}$$

encoded by elements of $\mathrm{Hom}_{\mathcal{F}}(1^{\oplus n}, 1^{\oplus m})$.

Example 1.15.

(i) The category of sets is algebraic with

$$\mathrm{Set} \simeq \mathrm{Fun}_{\Pi}(\mathrm{Fin}^{\mathrm{op}}, \mathrm{Set})$$

where Fin is the category of finite sets; there are no nontrivial operations in this case.

(ii) For a commutative ring R the category Mod_R of R-modules is algebraic with

$$\mathrm{Mod}_R \simeq \mathrm{Fun}_{\Pi}(\mathrm{FFree}_R^{\mathrm{op}}, \mathrm{Set})$$

where FFree_R is the category of finitely generated free R-modules. Operations are encoded by elements of $\mathrm{Hom}(R^{\oplus m}, R^{\oplus n})$, which are $(n \times m)$-matrices with values in R.

(iii) The category CAlg_R of commutative R-algebras is algebraic with

$$\mathrm{CAlg}_R \simeq \mathrm{Fun}_{\Pi}(\mathrm{Poly}_R^{\mathrm{op}}, \mathrm{Set})$$

where Poly_R is the category of finitely generated polynomial R-algebras $R[t_1, \dots, t_n]$, $n \geqslant 0$. Operations are encoded by elements of $\mathrm{Hom}(R[t_1, \dots, t_m], R[t_1, \dots, t_n])$, which are collections of polynomials with coefficients in R.

Definition 1.16. Let $\mathcal{C} \simeq \mathrm{Fun}_{\Pi}(\mathcal{F}^{\mathrm{op}}, \mathrm{Set})$ be an algebraic category. The *animation* of $\mathcal{C}$ is the ∞-category

$$\mathrm{Anim}(\mathcal{C}) \coloneqq \mathrm{Fun}_{\Pi}(\mathcal{F}^{\mathrm{op}}, \mathrm{Grpd}_\infty)$$

of functors $\mathcal{F}^{\mathrm{op}} \to \mathrm{Grpd}_\infty$ that send finite coproducts in $\mathcal{F}$ to finite products.

Remark 1.17. Historically, animation was called the *nonabelian derived category*. Since we also want to apply it to abelian contexts like Mod_R, and since we want the terminology to differentiate between the connective (bounded on the right) and nonconnective (unbounded) derived categories, we use this newer terminology from [CS].

Example 1.18. The ∞-category Grpd_∞ is the animation of the category of sets. Indeed, a functor $F \in \mathrm{Fun}_{\Pi}(\mathrm{Fin}^{\mathrm{op}}, \mathrm{Grpd}_\infty)$ is completely determined by the ∞-groupoid $F(\{*\})$ (because of the subscript $_{\Pi}$).

Example 1.19. For R a commutative ring, we denote the animation $\mathrm{Anim}(\mathrm{Mod}_R)$ by $\mathrm{D}(R)_{\geqslant 0}$ (or $\mathrm{D}(R)^{\leqslant 0}$). This is an ∞-category version of the right-bounded derived category: that is, it is equivalent to the ∞-categorical localization at quasi-isomorphisms of chain complexes with $H_i = H^{-i} = 0$ for $i < 0$.

Example 1.20. We denote the animation $\mathrm{Anim}(\mathrm{CAlg}_R)$ by dCAlg_R, and $\mathrm{dCRing} \coloneqq \mathrm{dCAlg}_{\mathbf{Z}} \simeq \mathrm{Anim}(\mathrm{CRing})$. We call the objects *derived commutative R-algebras* and derived commutative rings, respectively. There are also strict models in this case: dCAlg_R is equivalent to the ∞-categorical localization at weak homotopy equivalences of simplicial commutative R-algebras; if moreover R is a $\mathbf{Q}$-algebra, it is equivalent to the ∞-categorical localization at quasi-isomorphisms of commutative dg-R-algebras (with $H_i = H^{-i} = 0$ for $i < 0$).

Definition 1.21. An animated object $X \in \mathrm{Anim}(\mathcal{C})$ is *discrete* if the functor $X : \mathcal{F}^{\mathrm{op}} \to \mathrm{Grpd}_\infty$ takes values in sets. The inclusion $\mathrm{Set} \hookrightarrow \mathrm{Grpd}_\infty$, regarding sets as discrete ∞-groupoids, induces a fully faithful functor $\mathcal{C} \hookrightarrow \mathrm{Anim}(\mathcal{C})$ which identifies $\mathcal{C}$ with the discrete objects of $\mathrm{Anim}(\mathcal{C})$. There is a left adjoint $\pi_0 : \mathrm{Anim}(\mathcal{C}) \to \mathcal{C}$ sending

$$(\mathcal{F}^{\mathrm{op}} \to \mathrm{Grpd}_\infty) \mapsto (\mathcal{F}^{\mathrm{op}} \to \mathrm{Grpd}_\infty \xrightarrow{\pi_0} \mathrm{Set})$$

where $\pi_0 : \mathrm{Grpd}_\infty \to \mathrm{Set}$ sends a ∞-groupoid to its set of isomorphism classes of objects (or connected components).

Definition 1.22. More generally, we say that $X \in \mathrm{Anim}(\mathcal{C})$ is *n-truncated* if the functor $X : \mathcal{F}^{\mathrm{op}} \to \mathrm{Grpd}_\infty$ factors through n-truncated ∞-groupoids. If $\mathcal{F}$ is generated by $1 \in \mathcal{F}$ as in Remark 1.14, this is equivalent to the condition that $X^\circ := X(1)$ is n-truncated (has $\pi_i = 0$ for all $i > n$).

Remark 1.23. An animated R-module $M \in \mathrm{D}(R)_{\geqslant 0}$ amounts to the following data:

(i) For every integer $n \geqslant 0$, an ∞-groupoid $M_n \in \mathrm{Grpd}_\infty$.
(ii) For every R-linear map $\phi : R^{\oplus n} \to R^{\oplus m}$ (or $\phi \in \mathrm{Mat}_{m\times n}(R)$), a map $M_\phi : M_m \to M_n$.
(iii) For every two R-linear maps $\phi : R^{\oplus n} \to R^{\oplus m}$ and $\psi : R^{\oplus m} \to R^{\oplus l}$, a homotopy $h_{\phi,\psi} : M_\phi \circ M_\psi \simeq M_{\psi\circ\phi}$ of maps $M_l \to M_n$.
(iv) For every three R-linear maps ϕ, ψ, ω, a tetrahedron-shaped diagram expressing a "higher" homotopy between the homotopies $h_{\phi,\psi}$, $h_{\psi,\omega}$, $h_{\phi,\omega\circ\psi}$, and $h_{\psi\circ\phi,\omega}$.
(v) ...

This data is subject to the condition that the canonical map $M_n \to (M_1)^{\times n}$ is invertible for every $n \geqslant 0$ (in particular, $M_0 \simeq \mathrm{pt}$). We summarize (iii) and (iv) by saying that the maps M_ϕ are functorial *up to coherent homotopy.*

In particular, this data encodes:

(i) The underlying ∞-groupoid $M^\circ := M_1 \in \mathrm{Grpd}_\infty$.
(ii) Operations $(M^\circ)^{\times n} \to M^\circ$ on M°, for every $\phi \in \mathrm{Mat}_{n\times 1}(R)$. In particular, an addition operation $\mathrm{add} : M^\circ \times M^\circ \to M^\circ$.
(iii) An action of R on M°, i.e., a map $R \to \mathrm{End}(M^\circ)$ given by

$$R \simeq \mathrm{Mat}_{1\times 1}(R) = \mathrm{Hom}_{\mathcal{F}_R}(1,1) \xrightarrow{M} \mathrm{Maps}_{\mathrm{Grpd}_\infty}(M_1, M_1) = \mathrm{End}(M^\circ).$$

The endomorphism induced by $a \in R$ is the operation encoded by the matrix $a \in \mathrm{Mat}_{1\times 1}(R)$.
(iv) Associativity up to coherent homotopy. For example, given three points $x, y, z \in M^\circ$ we have a homotopy

$$\mathrm{add}(\mathrm{add}(x,y),z) \simeq \mathrm{add}(x,\mathrm{add}(y,z)).$$

Diagrammatically,

$$\begin{array}{ccc} M^\circ \times M^\circ \times M^\circ & \xrightarrow{\text{add}\times\text{id}} & M^\circ \times M^\circ \\ \downarrow{\scriptstyle \text{id}\times\text{add}} & & \downarrow{\scriptstyle \text{add}} \\ M^\circ \times M^\circ & \xrightarrow{\text{add}} & M^\circ. \end{array}$$

Informally speaking, we can think of an animated R-module as an ∞-groupoid equipped with a homotopy coherent R-module structure.

1.3. **Derived functors.**

Proposition 1.24 (Universal property)**.** *Let $\mathcal{C} \simeq \mathrm{Fun}_\Pi(\mathcal{F}^{\mathrm{op}}, \mathrm{Set})$ be an algebraic category.*

(i) *The category $\mathcal{C}$ is freely generated by $\mathcal{F}$ under filtered colimits ("unions") and reflexive coequalizers ("quotients by equivalence relations").*

(ii) *The ∞-category $\mathrm{Anim}(\mathcal{C})$ is freely generated by $\mathcal{F}$ under filtered colimits and geometric realizations ("derived quotients by equivalence relations"). More precisely, for every ∞-category $\mathcal{D}$ admitting filtered colimits and geometric realizations, the canonical functor*

$$\mathrm{Fun}_{\mathrm{filt},\boldsymbol{\Delta}}(\mathrm{Anim}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{F}, \mathcal{D})$$

is an equivalence, where the source is the ∞-category of functors that preserve filtered colimits and geometric realizations.

Definition 1.25. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between algebraic categories. Write $\mathcal{C} \simeq \mathrm{Fun}_\Pi(\mathcal{F}^{\mathrm{op}}, \mathrm{Set})$ and consider the restriction $F|_{\mathcal{F}} : \mathcal{F} \to \mathcal{D} \hookrightarrow \mathrm{Anim}(\mathcal{D})$. Applying the universal property of animation we obtain a unique functor

$$F^{\mathrm{anim}} : \mathrm{Anim}(\mathcal{C}) \to \mathrm{Anim}(\mathcal{D})$$

which preserves filtered colimits and geometric realizations, and sends $F^{\mathrm{anim}}(X) = F(X)$ if $X \in \mathcal{F}$. Moreover, it satisfies $\pi_0(F^{\mathrm{anim}}(X)) \simeq F(\pi_0(X))$ for any $X \in \mathrm{Anim}(\mathcal{C})$. We call F^{anim} the *animation* of F, and think of this as a left-derived functor of F.

Example 1.26. If $R \to S$ is a ring homomorphism, extension of scalars defines a functor $(-) \otimes_R S : \mathrm{Mod}_R \to \mathrm{Mod}_S$, whose animation we denote

$$(-) \otimes^{\mathbf{L}}_R S : \mathrm{D}(R)_{\geqslant 0} \to \mathrm{D}(S)_{\geqslant 0}.$$

Similarly, restriction of scalars $\mathrm{Mod}_S \to \mathrm{Mod}_S$ animates to a functor $\mathrm{D}(S)_{\geqslant 0} \to \mathrm{D}(R)_{\geqslant 0}$, right adjoint to $(-) \otimes^{\mathbf{L}}_R S$. If $M \in \mathrm{Mod}_R$ is a flat R-module, then we have

$$M \otimes^{\mathbf{L}}_R S \simeq M \otimes_R S$$

since by Lazard's theorem M can be written as a filtered colimit of finitely generated free modules, and $(-) \otimes^{\mathbf{L}}_R S$ commutes with filtered colimits.

1.4. **Cotangent complex.**

Remark 1.27. Let R be a commutative ring. Relative algebraic Kähler differentials define a canonical functor

$$\Omega_{-/R} : A \in \mathrm{CAlg}_R \quad \mapsto \quad (A, \Omega_{A/R}) \in \mathrm{CAlgMod}_R \tag{1.28}$$

where the target $\mathrm{CAlgMod}_R$ is the category of pairs (A, M) with $A \in \mathrm{CAlg}_R$ and $M \in \mathrm{Mod}_A$; a morphism $(A, M) \to (A', M')$ is an R-algebra homomorphism $A \to A'$ together with an A'-module homomorphism $M \otimes_A A' \to M'$. This functor is a section of the projection $\pi : \mathrm{CAlgMod}_R \to \mathrm{CAlg}_R$, $(A, M) \mapsto A$.

Construction 1.29. The animation of (1.28) is a functor

$$\Omega^{\mathrm{anim}}_{-/R} : \mathrm{dCAlg}_R \to \mathrm{Anim}(\mathrm{CAlgMod}_R)$$

which is a section of the projection $\pi^{\mathrm{anim}} : \mathrm{Anim}(\mathrm{CAlgMod}_R) \to \mathrm{Anim}(\mathrm{CAlg}_R)$. Thus the image of $A \in \mathrm{dCAlg}_R$ may be written as a pair $(A, \mathbf{L}_{A/R})$ with $\mathbf{L}_{A/R} \in \mathrm{D}(A)_{\geqslant 0}$, where $\mathrm{D}(A)_{\geqslant 0}$ is by definition the fibre of π^{anim} over A. (If A is discrete, this is consistent with our previous definition of $\mathrm{D}(A)_{\geqslant 0}$.) We call $\mathbf{L}_{A/R}$ *the (relative) cotangent complex of* A.

Remark 1.30. In Construction 1.29 we used the fact that $\mathrm{CAlgMod}_R$ is algebraic: $\mathcal{F}$ in this case is the full subcategory of pairs (A, M) where $A \in \mathrm{Poly}_R$ and $M \in \mathrm{FFree}_R$.

Proposition 1.31.

(i) *For every* $A \in \mathrm{dCAlg}_R$ *there is a canonical isomorphism* $\pi_0 \mathbf{L}_{A/R} \simeq \Omega_{\pi_0(A)/R}$.

(ii) *Let* $A \to B$ *be a morphism in* $\mathrm{Anim}(\mathrm{CAlg}_R)$. *Then there is an exact triangle*

$$\mathbf{L}_{A/R} \otimes^{\mathbf{L}}_A B \to \mathbf{L}_{B/R} \to \mathbf{L}_{B/A}$$

in $\mathrm{D}(B)$.

(iii) *Given morphisms* $A \to B$ *and* $A \to A'$ *in* dCAlg_R, *there is a canonical isomorphism*

$$\mathbf{L}_{B/A} \otimes^{\mathbf{L}}_B B' \simeq \mathbf{L}_{A' \otimes^{\mathbf{L}}_A B/A'}$$

where $B' \coloneqq B \otimes^{\mathbf{L}}_A A'$.

The cotangent complex corepresents derived derivations:

Proposition 1.32 (Universal property)**.** *For every* $A \in \mathrm{dCAlg}_R$ *there are canonical isomorphisms, functorial in* $M \in \mathrm{D}(R)_{\geqslant 0}$,

$$\mathrm{Maps}_{\mathrm{D}(A)_{\geqslant 0}}(\mathbf{L}_{A/R}, M) \simeq \mathrm{Fib}(\mathrm{Maps}_{\mathrm{dCAlg}_R}(A, A \oplus M) \to \mathrm{Maps}_{\mathrm{dCAlg}_R}(A, A))$$

where the fibre is taken over the identity map id_A.

2. Derived stacks

2.1. Sheaves.

Definition/Proposition 2.1. *Let $A \to B$ be a morphism of derived commutative rings. We say that it is* flat *if it satisfies the following equivalent conditions:*

(i) *The functor $(-) \otimes^{\mathbf{L}}_A B : \mathrm{D}(A)_{\geqslant 0} \to \mathrm{D}(B)_{\geqslant 0}$ is left-exact, i.e., it preserves discrete objects.*

(ii) *The induced ring homomorphism $\pi_0(A) \to \pi_0(B)$ is flat, and the canonical homomorphisms*
$$\pi_i(A) \otimes_{\pi_0(A)} \pi_0(B) \to \pi_i(B)$$
are bijective for all $i \geqslant 0$.

Definition 2.2. We extend the flat topology on $\mathrm{CRing}^{\mathrm{op}}$ to $\mathrm{dCRing}^{\mathrm{op}}$ as follows: we say that a morphism $A \to B$ in dCRing is *faithfully flat* if it is flat in the sense of Definition 2.1, and the induced ring homomorphism $\pi_0(A) \to \pi_0(B)$ is faithfully flat.

Definition 2.3. We say that $A \to B$ is *étale*, resp. *smooth*, if it is flat and the induced ring homomorphism $\pi_0(A) \to \pi_0(B)$ is étale, resp. smooth (in the sense of ordinary commutative algebra). One can show that this is equivalent to the following condition: $\pi_0(A) \to \pi_0(B)$ is of finite presentation, and the relative cotangent complex $\mathbf{L}_{B/A}$ is zero, resp. of Tor-amplitude $[0, 0]$.

Definition 2.4. Let $\mathcal{V}$ be an ∞-category. A functor $F : \mathrm{dCRing} \to \mathcal{V}$ is a *sheaf* (for the flat, resp. étale, topology) if it satisfies the following conditions:

(i) F preserves finite products. That is, for every finite collection $(A_i)_i$ of derived commutative rings, the canonical morphism
$$F(\prod_i A_i) \to \prod_i F(A_i)$$
is invertible.

(ii) For every faithfully flat (resp. étale and faithfully flat) morphism $A \to B$, the diagram
$$F(A) \to F(B) \rightrightarrows F(B \otimes^{\mathbf{L}}_A B) \substack{\to\\\to\\\to} F(B \otimes^{\mathbf{L}}_A B \otimes^{\mathbf{L}}_A B) \substack{\to\\\to\\\to\\\to} \cdots$$
is a limit diagram. In other words, $F(A)$ is isomorphic to the totalization $\mathrm{Tot}(F(B^\bullet))$ of the cosimplicial diagram $B^\bullet = B^{\otimes \bullet+1}$ (tensor product over A).

Remark 2.5. The above definition makes sense more generally for a presheaf $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{V}$ where $\mathcal{C}$ is any ∞-category with a Grothendieck topology. We write $\mathrm{Shv}(\mathcal{C}; \mathcal{V})$ for the full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{V})$ spanned by sheaves.

2.2. Derived stacks.

Definition 2.6. Let R be a commutative ring. A *derived stack* over R is a functor $X : \mathrm{dCAlg}_R \to \mathrm{Grpd}_\infty$ satisfying étale descent. We denote the ∞-category of derived stacks by $\mathrm{dStk} = \mathrm{Shv}(\mathrm{dCAlg}_R^{\mathrm{op}}; \mathrm{Grpd}_\infty)$.

Example 2.7. An *affine derived scheme* over R is the functor $\mathrm{Spec}(A) : \mathrm{dCAlg}_R \to \mathrm{Grpd}_\infty$, $B \mapsto \mathrm{Maps}_{\mathrm{dCAlg}_R}(A, B)$, corepresented by an animated algebra $A \in \mathrm{dCAlg}_R$.

Definition 2.8. Given a derived stack X, the restriction of the functor $X : \mathrm{dCAlg}_R \to \mathrm{Grpd}_\infty$ along $\mathrm{CAlg}_R \hookrightarrow \mathrm{dCAlg}_R$ is called the *classical truncation* of X and is denoted X_{cl}. For example, $\mathrm{Spec}(A)_{\mathrm{cl}} \simeq \mathrm{Spec}(\pi_0(A))$ for any $A \in \mathrm{dCAlg}_R$. In general, X is a *higher stack* in the sense of [HS]. If the ∞-groupoid $X(A)$ is 1-truncated for every $A \in \mathrm{CAlg}_R$, then $X_{\mathrm{cl}} : \mathrm{CAlg}_R \to \mathrm{Grpd}$ is a 1-stack. For example, this will be the case if X is 1-Artin.

Definition 2.9. Let $j : U \to X$ be a morphism of derived stacks.

(i) If X and U are affine, we say j is an open immersion if it is étale ($\mathcal{O}_X \to \mathcal{O}_U$ is an étale morphism of derived commutative rings) and $U_{\mathrm{cl}} \to X_{\mathrm{cl}}$ is an open immersion of classical affines.

(ii) If X is affine, we say j is an open immersion if it is a monomorphism (the diagonal $U \to U \times_X U$ is an isomorphism) and there exists a collection of affines $(U_\alpha)_\alpha$ and a surjection $\coprod_\alpha U_\alpha \twoheadrightarrow U$ such that each composite $U_\alpha \to X$ is an open immersion of affines.

(iii) In general, j is an open immersion if for every affine S and every morphism $S \to X$, the base change $U \times_X S \to S$ is an open immersion to an affine.

Definition 2.10 (Derived schemes)**.** A derived stack X is a *derived scheme* if there exists a collection $(U_\alpha \hookrightarrow X)_\alpha$ of open immersions where U_α are affine derived schemes, and the induced morphism $\coprod_\alpha U_\alpha \twoheadrightarrow X$ is surjective. A morphism $f : X \to Y$ is *schematic* if for every affine V and every morphism $V \to Y$, the fibre $X \times^{\mathbf{R}}_Y V$ is a derived scheme. A schematic morphism $f : X \to Y$ of derived stacks is *smooth*, resp. *étale*, if for every affine V and every morphism $V \to Y$, there exists a collection of open immersions $(U_\alpha \hookrightarrow X \times_Y V)_\alpha$ where each U_α is affine, the morphism $\coprod_\alpha U_\alpha \to X \times_Y V$ is surjective, and each composite $U_\alpha \to X \times_Y V \to V$ is a smooth, resp. étale, morphism of affines.

Remark 2.11. A derived scheme X is 0-truncated, in the sense that on (ordinary) commutative R-algebras, the functor $X : \mathrm{dCAlg}_R \to \mathrm{Grpd}_\infty$ takes values in sets (= 0-truncated or discrete ∞-groupoids).

2.3. **Derived ∞-categories of quasi-coherent sheaves.**

Definition 2.12. Let A be a derived commutative ring. In the ∞-category $\mathrm{D}(A)_{\geqslant 0}$, or more generally any ∞-category with a zero object and finite co/limits, we define the *suspension* and *loop space* functors Σ and Ω by the cocartesian and cartesian squares

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma(X), \end{array} \qquad \begin{array}{ccc} \Omega(X) & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & X, \end{array}$$

respectively. These form an adjoint pair (Σ, Ω). We define the ∞-category $\mathrm{D}(A)$, the (unbounded) derived ∞-category of A, by forcing these functors to become mutually inverse; that is, we take the limit

$$\mathrm{D}(A) \to \cdots \xrightarrow{\Omega} \mathrm{D}(A)_{\geqslant 0} \xrightarrow{\Omega} \mathrm{D}(A)_{\geqslant 0}.$$

In the resulting ∞-category, we have $M \simeq \Omega\Sigma(M) \simeq \Sigma\Omega(M)$ for all $M \in \mathrm{D}(A)$. This property means that $\mathrm{D}(A)$ is a *stable ∞-category*. Moreover, it has a t-structure $(\mathrm{D}(A)_{\geqslant 0}, \mathrm{D}(A)_{<0})$. We set $M[1] := \Sigma(M)$ and $M[-1] := \Omega(M)$ for any $M \in \mathrm{D}(A)$.

Definition 2.13. A derived A-module $M \in \mathrm{D}(A)$ is *perfect* if it belongs to the thick subcategory generated by A; i.e., if it can be built out of A using finite limits, finite colimits, and direct summands. We write $\mathrm{Perf}(A) \subseteq \mathrm{D}(A)$ for the full subcategory of perfect A-modules.

Theorem 2.14 (Lurie, Toën)**.** *The functor*

$$\mathrm{dCAlg}_R \to \mathrm{Cat}_\infty, \quad A \mapsto \mathrm{D}(A)$$

is a sheaf for the flat topology. The same holds for $A \mapsto \mathrm{D}(A)_{\geqslant 0}$ *and* $A \mapsto \mathrm{Perf}(A)$.

Definition 2.15. Let X be a derived stack. The derived ∞-category of quasi-coherent sheaves $\mathrm{D}_{\mathrm{qc}}(X)$ is the limit

$$\mathrm{D}_{\mathrm{qc}}(X) := \varprojlim_{(A,x)} \mathrm{D}(A)$$

over the category of pairs (A, x) where $A \in \mathrm{dCAlg}_R$ and $x \in X(A)$. Given a morphism of pairs $(A, x) \to (A', x')$, i.e., an R-algebra homomorphism $A \to A'$ and a homotopy $x|_{A'} \simeq x' \in X(A')$, the transition functor $\mathrm{D}(A) \to \mathrm{D}(A')$ is derived extension of scalars.

In other words,

$$\mathrm{D}_{\mathrm{qc}} : \mathrm{dStk}^{\mathrm{op}} \to \mathrm{Cat}_\infty$$

is the right Kan extension of the presheaf $\mathrm{Spec}(A) \mapsto \mathrm{D}(A)$ along the inclusion $\mathrm{Aff} \hookrightarrow \mathrm{dStk}$. We refer to objects of $\mathrm{D}_{\mathrm{qc}}(X)$ as *quasi-coherent complexes* on X. Note that $\mathrm{D}_{\mathrm{qc}}(X)$ is stable, since this property is preserved under formation of limits.

Remark 2.16. Recall that, by one characterization of right Kan extensions, the presheaf $\mathrm{D}_{\mathrm{qc}} : \mathrm{dStk}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ is the unique limit-preserving functor extending $\mathrm{D}_{\mathrm{qc}} : \mathrm{Aff}^{\mathrm{op}} \to \mathrm{Cat}_\infty$, $\mathrm{Spec}(A) \mapsto \mathrm{D}_{\mathrm{qc}}(\mathrm{Spec}(A)) \simeq \mathrm{D}(A)$. In particular, it sends colimits of derived stacks to limits of ∞-categories.

Example 2.17. Let G be a group scheme over R, acting on a derived stack X. The *quotient stack* $[X/G]$ is the colimit of the action groupoid

$$X_\bullet = [\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} G \times X \rightrightarrows X].$$

Then there is a limit diagram of ∞-categories

$$\mathrm{QCoh}([X/G]) \to \mathrm{QCoh}(X) \rightrightarrows \mathrm{QCoh}(G \times X) \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathrm{QCoh}(G \times G \times X) \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots.$$

In other words, the canonical functor $\mathrm{QCoh}([X/G]) \to \mathrm{Tot}(\mathrm{QCoh}(X_\bullet))$ is an equivalence, where $X_\bullet = [\cdots \substack{\to\\\to\\\to} G \times X \rightrightarrows X]$ is the action groupoid (whose colimit is the quotient stack $[X/G]$). We call

$$\mathrm{QCoh}^G(X) \coloneqq \mathrm{Tot}(\mathrm{QCoh}(X_\bullet))$$

the *G-equivariant derived ∞-category of quasi-coherent sheaves on X*; its objects, which we call *G-equivariant quasi-coherent sheaves* on X, are quasi-coherent sheaves $\mathcal{F}$ on X together with a (specified) isomorphism $\mathrm{act}^*\mathcal{F} \simeq \mathrm{pr}^*\mathcal{F}$ on $G \times X$, as well as a homotopy coherent system of isomorphisms on the higher terms $G^{\times n} \times X$. Thus Example 2.17 says that quasi-coherent sheaves on $[X/G]$ are quasi-coherent sheaves on X that are G-equivariant in the homotopy coherent sense.

2.4. **Cotangent complexes.**

Definition 2.18. Let $X : \mathrm{dCAlg}_R \to \mathrm{Grpd}_\infty$ be a derived stack. We say that *X admits a cotangent complex* $\mathbf{L}_{X/R}$ if and only if the following conditions hold:

(i) For every $A \in \mathrm{dCAlg}_R$ and every $x \in X(A)$, denote by $F_x(N)$ the fibre at x of the map
$$X(A \oplus N) \to X(A)$$
for every $N \in \mathrm{D}(A)_{\geqslant 0}$. Then the functor $F_x(-)$ is corepresented by a derived A-module M_x which is eventually connective, i.e., $M_x[n] \in \mathrm{D}(A)_{\geqslant 0}$ for some n.

(ii) For every morphism $A \to B$ in dCAlg_R and every $N \in \mathrm{D}(B)_{\geqslant 0}$, the commutative square
$$\begin{array}{ccc} X(A \oplus N) & \longrightarrow & X(B \oplus N) \\ \downarrow & & \downarrow \\ X(A) & \longrightarrow & X(B) \end{array}$$
is cartesian.

Note that under these conditions, there exists an object $\mathbf{L}_{X/R} \in \mathrm{D}_{\mathrm{qc}}(X)$ such that $x^*\mathbf{L}_{X/R} \simeq M_x$ for every $A \in \mathrm{dCAlg}_R$ and $x \in X(A)$ (modulo the equivalence $\mathrm{D}_{\mathrm{qc}}(\mathrm{Spec}(A)) \simeq \mathrm{D}(A)$). We will often write $\mathbf{L}_X \coloneqq \mathbf{L}_{X/R}$.

Remark 2.19. As in the affine case (the case of derived commutative rings), it is possible to talk about *relative* cotangent complexes $\mathbf{L}_{X/Y}$ for morphisms $f : X \to Y$. See [Kha3, Def. 8.31], [Toë, §5.3], or [TV2, §1.4.1]. We have $\mathbf{L}_{X/\mathrm{Spec}(R)} \simeq \mathbf{L}_{X/R}$, and by Theorem 2.20 below, there is an exact triangle

$$f^*\mathbf{L}_Y \to \mathbf{L}_X \to \mathbf{L}_{X/Y}$$

in the stable ∞-category $\mathrm{D}_{\mathrm{qc}}(X)$.

Theorem 1.31 immediately globalizes as follows:

Theorem 2.20.

(i) *Let S be a derived stack and $f : X \to Y$ a morphism over S. If X and Y admit cotangent complexes over S, then f admits a relative cotangent complex such that there is an exact triangle*

$$f^*\mathbf{L}_{Y/S} \to \mathbf{L}_{X/S} \to \mathbf{L}_{X/Y}$$

in $\mathrm{D}_{\mathrm{qc}}(X)$.

(ii) *Let $f : X \to Y$ be a morphism of derived stacks. If f admits a relative cotangent complex, then for every morphism $Y' \to Y$ the derived base change $X \times^{\mathbf{R}}_Y Y' \to Y'$ admits a relative cotangent complex, and moreover there is a canonical isomorphism*

$$p^*\mathbf{L}_{X/Y} \simeq \mathbf{L}_{X \times^{\mathbf{R}}_Y Y'/Y'}$$

in $\mathrm{D}_{\mathrm{qc}}(X \times^{\mathbf{R}}_Y Y')$, *where $p : X \times^{\mathbf{R}}_Y Y' \to X$ is the projection.*

Example 2.21. Let G be a smooth group scheme over R, and let U be a derived stack over R with a G-action. Assuming that U admits a cotangent complex $\mathbf{L}_U$, we can compute the cotangent complex of the quotient stack $[U/G]$ as follows. Consider the cartesian square:

$$\begin{array}{ccc} G \times U & \xrightarrow{\mathrm{pr}} & U \\ \downarrow{\scriptstyle \mathrm{act}} & & \downarrow{\scriptstyle p} \\ U & \xrightarrow{p} & [U/G]. \end{array}$$

We have

$$\mathbf{L}_{U/[U/G]} \simeq d^*\mathrm{act}^*\mathbf{L}_{U/[U/G]} \simeq d^*\mathbf{L}_{G\times U/U} \simeq d^*\mathrm{pr}_1^*\mathbf{L}_G$$

where $d = (e, \mathrm{id}) : U \to G \times U$ and $\mathrm{pr}_1 : G \times U \to G$, using Theorem 2.20(ii) twice. Since $\mathrm{pr}_1 \circ d$ factors as the projection $f : U \to \mathrm{Spec}(R)$ followed by the identity section $e : \mathrm{Spec}(R) \to G$, we get

$$\mathbf{L}_{U/[U/G]} \simeq f^*e^*\mathbf{L}_G \simeq f^*\mathfrak{g}^\vee$$

where $\mathfrak{g}^\vee = e^*\mathbf{L}_G \simeq e^*\Omega_G$ is the dual Lie algebra of G (recall that G is smooth over R). Finally, we have by Theorem 2.20(i) an exact triangle

$$p^*\mathbf{L}_{[U/G]} \to \mathbf{L}_U \to \mathbf{L}_{U/[U/G]}$$

where $p : U \twoheadrightarrow [U/G]$ is the quotient morphism. Under the equivalence $\mathrm{D}_{\mathrm{qc}}([U/G]) \simeq \mathrm{D}^G_{\mathrm{qc}}(U)$, $\mathbf{L}_{[U/G]}$ may be regarded as the quasi-coherent complex

$$\mathrm{Fib}(\mathbf{L}_U \to f^*\mathfrak{g}^\vee) \in \mathrm{D}_{\mathrm{qc}}(U)$$

with the natural G-action induced by the action on U. For example, if U is a smooth scheme, then this is a 2-term complex with Ω_U in degree 0 and $f^*\mathfrak{g}^\vee$ in (homological) degree -1. Note that if G is finite (and hence étale), we have $\mathfrak{g}^\vee \simeq 0$.

2.5. **Moduli of sheaves.**

Definition 2.22. [TV1] Let $\mathcal{M}_{\mathrm{Perf}}$ denote the functor

$$\mathrm{dCAlg}_R \to \mathrm{Grpd}_\infty, \quad A \mapsto \mathrm{Perf}(A)^{\simeq}$$

sending A to the ∞-groupoid of perfect derived A-modules. The superscript $^{\simeq}$ indicates that we take the underlying ∞-groupoid, obtained by discarding all non-invertible morphisms. By Theorem 2.14, this satisfies étale descent and hence defines a derived stack over R. By Yoneda, there is a *universal perfect complex*

$$\mathcal{E}^{\mathrm{univ}} \in \mathrm{Perf}(\mathcal{M}_{\mathrm{Perf}})$$

such that for every derived stack X and every perfect complex $\mathcal{E} \in \mathrm{Perf}(X)$, there is a unique morphism $f : X \to \mathcal{M}_{\mathrm{Perf}}$ together with an isomorphism $f^*(\mathcal{E}^{\mathrm{univ}}) \simeq \mathcal{E}$.

Remark 2.23. We can similarly consider the stacks $\mathcal{M}_{\mathrm{D}_{\mathrm{coh}}}$ and $\mathcal{M}_{\mathrm{D}_{\mathrm{pscoh}}}$ sending $A \in \mathrm{dCRing}_R$ to $\mathrm{D}_{\mathrm{coh}}(A)^{\simeq}$ or $\mathrm{D}_{\mathrm{pscoh}}(A)^{\simeq}$, respectively. Here $\mathrm{D}_{\mathrm{pscoh}}(A) \subseteq \mathrm{D}_{\mathrm{qc}}(A)$ is the full subcategory of *pseudocoherent* derived A-modules (sometimes called *almost perfect* A-modules) and $\mathrm{D}_{\mathrm{coh}}(A) \subseteq \mathrm{D}_{\mathrm{pscoh}}(A)$ is the full subcategory of *coherent* derived A-modules.[1] The inclusions[2] $\mathrm{Perf}(A) \subseteq \mathrm{D}_{\mathrm{pscoh}}(A)$ and $\mathrm{D}_{\mathrm{coh}}(A) \subseteq \mathrm{D}_{\mathrm{pscoh}}(A)$ induce open immersions of derived stacks

$$\mathcal{M}_{\mathrm{Perf}} \hookrightarrow \mathcal{M}_{\mathrm{D}_{\mathrm{pscoh}}}, \quad \mathcal{M}_{\mathrm{D}_{\mathrm{coh}}} \hookrightarrow \mathcal{M}_{\mathrm{D}_{\mathrm{pscoh}}},$$

but $\mathcal{M}_{\mathrm{D}_{\mathrm{coh}}}$ and $\mathcal{M}_{\mathrm{D}_{\mathrm{pscoh}}}$ do not admit cotangent complexes.

Theorem 2.24. *The perfect complex*

$$\mathbf{L}_{\mathcal{M}_{\mathrm{Perf}}} := \mathcal{E}^{\mathrm{univ}} \otimes^{\mathbf{L}} \mathcal{E}^{\mathrm{univ},\vee}[-1]$$

is a cotangent complex for the derived stack $\mathcal{M}_{\mathrm{Perf}}$.

Lemma 2.25. *Let $A \in \mathrm{dCAlg}_R$ and $M \in \mathrm{Perf}(A)$. For every $N \in \mathrm{Perf}(A)_{\geqslant 0}$ denote by $F_M(N)$ the fibre at M of the map of anima*

$$\mathrm{Perf}(A \oplus N)^{\simeq} \to \mathrm{Perf}(A)^{\simeq}$$

given by extending scalars along the canonical homomorphism $A \oplus N \to A$. Then we have canonical isomorphisms

$$F_M(N) \simeq \mathrm{Maps}_{\mathrm{D}(A)}(M \otimes^{\mathbf{L}}_A M^\vee[-1], N),$$

natural in N.

Proof. By definition, $F_M(N)$ is the ∞-groupoid of deformations of M along $A \oplus N \to A$, i.e., of pairs $(\widetilde{M}, \theta)$ where $\widetilde{M}$ is an $A \oplus N$-module and θ is an

[1]When A is noetherian these are defined as follows: $M \in \mathrm{D}(A)$ is pseudocoherent if it is eventually connective ($\pi_i(M) = 0$ for $i \ll 0$) and its homotopy groups $\pi_i(M)$ are finitely generated $\pi_0(A)$-modules (see [HA, Def. 7.2.4.10]); it is *coherent* if it is additionally eventually coconnective ($\pi_i(M) = 0$ for $i \gg 0$).

[2]Note that $\mathrm{Perf}(A) \subseteq \mathrm{D}_{\mathrm{coh}}(A)$ if and only if A is eventually coconnective.

A-linear isomorphism $\widetilde{M} \otimes^{\mathbf{L}}_{A\oplus N} A \simeq M$. Since the square

$$\begin{array}{ccc} A \oplus N & \longrightarrow & A \\ \downarrow & & \downarrow \\ A & \longrightarrow & A \oplus N[1] \end{array}$$

is cartesian, this is equivalent to the ∞-groupoid of deformations of $M \otimes^{\mathbf{L}}_A (A \oplus N[1]) \simeq M \oplus (M \otimes^{\mathbf{L}}_A N[1])$ along the trivial derivation $A \to A \oplus N[1]$. Equivalently, this is the ∞-groupoid of automorphisms of $M \oplus (M \otimes^{\mathbf{L}} N[1])$ over $A \oplus N[1]$ which extend to the identity $\mathrm{id}_M : M = M$ along $A \oplus N[1] \to A$. That is,

$$F_M(N) \simeq \mathrm{End}_{A\oplus N[1]}(M \oplus (M \otimes^{\mathbf{L}} N[1])) \underset{\mathrm{End}_{\mathrm{D}(A)}(M)}{\times} \{\mathrm{id}_M\}$$

where we can write End instead of Aut since every such endomorphism is necessarily invertible. Thus we have

$$\begin{aligned} F_M(N) &\simeq \mathrm{Maps}_{\mathrm{D}(A)}(M, (M \oplus (M \otimes^{\mathbf{L}} N[1])) \underset{M}{\times} 0) \\ &\simeq \mathrm{Maps}_{\mathrm{D}(A)}(M, M \otimes^{\mathbf{L}} N[1]) \simeq \mathrm{Maps}_{\mathrm{D}(A)}(M \otimes^{\mathbf{L}} M^\vee[-1], N) \end{aligned}$$

where the last isomorphism follows from the fact that M is perfect, hence dualizable. □

Proof of Theorem 2.24. Let $A \in \mathrm{dCAlg}_R$ and $x : \mathrm{Spec}(A) \to \mathcal{M}_{\mathrm{Perf}}$ an A-point classifying a perfect derived A-module $M \in \mathrm{Perf}(A)$. By Lemma 2.25, the ∞-groupoid of R-linear derivations with values in M is corepresented by $M \otimes^{\mathbf{L}} M^\vee[-1]$. As (A, x) varies, the perfect complexes $M \otimes^{\mathbf{L}} M^\vee[-1]$ assemble into the perfect complex $\mathcal{E}^{\mathrm{univ}} \otimes^{\mathbf{L}} \mathcal{E}^{\mathrm{univ},\vee}[-1] \in \mathrm{Perf}(\mathcal{M}_{\mathrm{Perf}})$, which is therefore a cotangent complex for $\mathcal{M}_{\mathrm{Perf}}$. □

Construction 2.26. Let $R \in \mathrm{dCRing}$ and let X and Y be derived stacks over R. The *derived mapping stack* $\underline{\mathrm{Maps}}(X, Y)$ is the functor

$$\underline{\mathrm{Maps}}(X, Y) : \mathrm{dCAlg}_R \to \mathrm{Grpd}_\infty, \quad A \mapsto \mathrm{Maps}_{\mathrm{Spec}(R)}(X \times \mathrm{Spec}(A), Y)$$

sending A to the ∞-groupoid of R-morphisms $X \times \mathrm{Spec}(A) \to Y$. More generally, we may form the derived mapping stack

$$\underline{\mathrm{Maps}}_S(X, Y) : \mathrm{dAff}^{\mathrm{op}}_{/S} \to \mathrm{Grpd}_\infty, \quad (\mathrm{Spec}(A) \to S) \mapsto \mathrm{Maps}_S(X \underset{S}{\times} \mathrm{Spec}(A), Y)$$

over a derived stack S, whenever X and Y are defined over S. There is an *evaluation morphism*

$$\mathrm{ev} : \underline{\mathrm{Maps}}_S(X, Y) \underset{S}{\times} X \to Y,$$

classified by the identity $\mathrm{id} : \underline{\mathrm{Maps}}_S(X, Y) \to \underline{\mathrm{Maps}}_S(X, Y)$.

Warning 2.27. By abuse of notation, all products are implicitly fibred over the $\mathrm{Spec}(R)$. For example, $X \times \mathrm{Spec}(A)$ really means the derived fibred product $X \times^{\mathbf{R}}_{\mathrm{Spec}(R)} \mathrm{Spec}(A)$, which need not agree with the classical fibred product if X is not flat over R. In particular, there is an identification

$$\underline{\mathrm{Maps}}(X, Y)_{\mathrm{cl}} \simeq \underline{\mathrm{Hom}}(X_{\mathrm{cl}}, Y_{\mathrm{cl}})$$

of the classical truncation of $\underline{\mathrm{Maps}}_R(X, Y)$ with the classical Hom stack when X is flat over R, but not in general.

Theorem 2.28. *Suppose X and Y are derived stacks over R. Set $H := \underline{\mathrm{Maps}}(X, Y)$ and consider the diagram*

$$H \xleftarrow{\pi} X \times H \xrightarrow{\mathrm{ev}} Y$$

where π is the projection. If X is a derived scheme, proper and of finite Tor-amplitude over R, and Y admits a cotangent complex $\mathbf{L}_Y$, then the perfect complex

$$\mathbf{L}_H \simeq \pi_*(\mathrm{ev}^*(\mathbf{L}_Y) \otimes (K_X \boxtimes \mathcal{O}_H))$$

is a cotangent complex for H. Here $K_X = p^!(\mathcal{O}_{\mathrm{Spec}(R)})$ denotes the dualizing complex, where $p : X \to \mathrm{Spec}(R)$ is the projection, and $K_X \boxtimes \mathcal{O}_H = \mathrm{pr}_1^*(K_X)$ where $\mathrm{pr}_1 : X \times H \to X$ is the projection.

Proof. Given $A \in \mathrm{dCAlg}_R$, an A-point $h \in H(A)$ classifying a morphism $f : X_R \to Y$, and $M \in \mathrm{D}(A)_{\geqslant 0}$, derivations of H at h are extensions of the morphism $f : X_A := X \times \mathrm{Spec}(A) \to Y$ along $X_A \hookrightarrow X_{A \oplus M}$. Since the latter can be regarded as the trivial square-zero extension of X_A by $p_A^*(M)$, where $p_A : X_A \to \mathrm{Spec}(A)$ is the projection, these are classified by the cotangent complex of Y, i.e.,

$$\mathrm{Der}_h(H, M) \simeq \mathrm{Maps}_{\mathrm{D}(X_A)}(f^*\mathbf{L}_Y, p_A^*(M)).$$

The assumptions on X imply that p_A^* admits a left adjoint $p_{A,\sharp} := p_{A,*}(- \otimes K_{X_A/A})$, hence $\mathrm{Der}_h(H, -)$ is corepresented by $p_{A,\sharp} f^* \mathbf{L}_Y$. Now

$$\mathbf{L}_H := \pi_\sharp \mathrm{ev}^*(\mathbf{L}_Y) := \pi_*(\mathrm{ev}^*(\mathbf{L}_Y) \otimes K_{X \times H/H}) \simeq \pi_*(\mathrm{ev}^*(\mathbf{L}_Y) \otimes (K_X \boxtimes \mathcal{O}_H))$$

is the unique perfect complex on H such that $h^*(\mathbf{L}_H) \simeq p_{A,\sharp} f^*(\mathbf{L}_Y)$, in view of the commutative diagram

$$\begin{array}{ccccc}
H & \xleftarrow{\pi} & X \times H & \xrightarrow{\mathrm{ev}} & Y \\
{\scriptstyle h}\uparrow & & \uparrow & & \uparrow{\scriptstyle f} \\
\mathrm{Spec}(A) & \xleftarrow{p_A} & X_A & = & X_A
\end{array}$$

where the left-hand square is cartesian. It follows that $\mathbf{L}_H$ is a cotangent complex for H. □

Definition 2.29. Let X be a smooth proper scheme over R.

(i) The *moduli stack of perfect complexes over X* is the derived mapping stack

$$\mathcal{M}_{\mathrm{Perf}(X)} = \underline{\mathrm{Maps}}(X, \mathcal{M}_{\mathrm{Perf}}).$$

For $A \in \mathrm{dCAlg}_R$, its A-points are morphisms $X_A := X \times \mathrm{Spec}(A) \to \mathcal{M}_{\mathrm{Perf}}$ over $\mathrm{Spec}(A)$, i.e., perfect complexes on X_A.

(ii) Given a group scheme G over R, the *moduli stack of G-torsors on X* (a.k.a. *principal G-bundles* on X) is the derived mapping stack

$$\mathcal{M}_{\mathrm{Bun}_G(X)} = \underline{\mathrm{Maps}}(X, BG).$$

For $A \in \mathrm{dCAlg}_R$, its A-points are morphisms $X_A \to BG$ over $\mathrm{Spec}(A)$, i.e., G-torsors on X_A.

(iii) The *moduli stack of vector bundles on* X is the substack $\mathcal{M}_{\mathrm{Vect}(X)} \subseteq \mathcal{M}_{\mathrm{Perf}(X)}$ defined as follows: for $A \in \mathrm{dCAlg}_R$, an A-point of $\mathcal{M}_{\mathrm{Perf}(X)}$ belongs to $\mathcal{M}_{\mathrm{Vect}(X)}$ if and only if the corresponding perfect complex $\mathcal{F} \in \mathrm{Perf}(X_A)$ is of Tor-amplitude $[0,0]$, i.e., it is connective and flat over X_A.

(iv) The *moduli stack of coherent sheaves over* X is the substack $\mathcal{M}_{\mathrm{Coh}(X)}$ of the derived mapping stack

$$\mathcal{M}_{\mathrm{D_{coh}}(X)} = \underline{\mathrm{Maps}}(X, \mathcal{M}_{\mathrm{D_{coh}}})$$

defined as follows: for $A \in \mathrm{dCAlg}_R$, an A-point of $\mathcal{M}_{\mathrm{D_{coh}}(X)}$ belongs to $\mathcal{M}_{\mathrm{Coh}(X)}$ if and only if the corresponding coherent complex $\mathcal{F} \in \mathrm{D_{coh}}(X_A)$ is connective and flat over $\mathrm{Spec}(A)$.

Remark 2.30. Since vector bundles are locally trivial, there is a canonical isomorphism of derived stacks

$$\mathcal{M}_{\mathrm{Vect}(X)} \simeq \coprod_{n \geqslant 0} \mathcal{M}_{\mathrm{Bun}_{\mathrm{GL}_n}(X)}.$$

Remark 2.31.

(i) The classical truncation of $\mathcal{M}_{\mathrm{Coh}(X)}$ is identified with the usual moduli stack of coherent sheaves on X. That is: if A is an ordinary R-algebra, the ∞-groupoid of A-points of $\mathcal{M}_{\mathrm{Coh}(X)}$ is equivalent to the 1-groupoid of coherent sheaves on X_A which are flat over $\mathrm{Spec}(A)$.

(ii) There is an inclusion $\mathcal{M}_{\mathrm{Coh}(X)} \subseteq \mathcal{M}_{\mathrm{Perf}(X)}$. Indeed, one can check that if a connective coherent complex $\mathcal{F} \in \mathrm{Coh}(X_R)$ is flat over $\mathrm{Spec}(R)$, where $R \in \mathrm{dCAlg}_k$, then $\mathcal{F}$ is perfect.

Combining Theorems 2.28 and 2.24, we have:

Corollary 2.32. *Let* X *be a smooth proper scheme over* R*. Then the derived stack* $\mathcal{M}_{\mathrm{Perf}(X)}$ *admits a relative cotangent complex*

$$\mathbf{L}_{\mathcal{M}_{\mathrm{Perf}(X)}} = \mathrm{pr}_{2,*}(\mathcal{E}_X \otimes^{\mathbf{L}} \mathcal{E}_X^\vee[1] \otimes^{\mathbf{L}} \mathrm{pr}_1^*(K_X))$$

where pr_i *are the two projections from* $X \times \mathcal{M}_{\mathrm{Perf}(X)}$*, and* $\mathcal{E}_X := \mathrm{ev}^*(\mathcal{E}^{\mathrm{univ}})$ *is the inverse image of the universal perfect complex along the evaluation morphism*

$$\mathrm{ev} : X \times \mathcal{M}_{\mathrm{Perf}(X)} \to \mathcal{M}_{\mathrm{Perf}}.$$

Remark 2.33. The inclusions $\mathcal{M}_{\mathrm{Vect}(X)} \hookrightarrow \mathcal{M}_{\mathrm{Coh}(X)}$ and $\mathcal{M}_{\mathrm{Coh}(X)} \hookrightarrow \mathcal{M}_{\mathrm{Perf}(X)}$ are open immersions. That is, for any $A \in \mathrm{dCAlg}_R$ and any morphism $\mathrm{Spec}(A) \to \mathcal{M}_{\mathrm{Coh}(X)}$, resp. $\mathrm{Spec}(A) \to \mathcal{M}_{\mathrm{Perf}(X)}$, the base changes

$$\mathcal{M}_{\mathrm{Vect}(X)} \overset{\mathbf{R}}{\underset{\mathcal{M}_{\mathrm{Coh}(X)}}{\times}} \mathrm{Spec}(A) \to \mathrm{Spec}(A),$$

$$\text{resp.} \quad \mathcal{M}_{\mathrm{Coh}(X)} \overset{\mathbf{R}}{\underset{\mathcal{M}_{\mathrm{Perf}(X)}}{\times}} \mathrm{Spec}(A) \to \mathrm{Spec}(A)$$

are open immersions of derived schemes. In particular, $\mathcal{M}_{\mathrm{Vect}(X)}$ and $\mathcal{M}_{\mathrm{Coh}(X)}$ admit cotangent complexes given by the same formula as in Corollary 2.32.

Similarly, combining Theorem 2.28 with our computation $\mathbf{L}_{BG} \simeq \mathfrak{g}^\vee[-1]$ (Example 2.21) yields:

Corollary 2.34. *Let X be a smooth proper scheme over R. Then the derived stack $\mathcal{M}_{\mathrm{Bun}_G(X)}$ admits a relative cotangent complex*

$$\mathbf{L}_{\mathcal{M}_{\mathrm{Bun}_G(X)}} = \mathrm{pr}_{2,*}(\mathrm{ev}^*\mathfrak{g}^\vee[-1] \otimes^{\mathbf{L}} \mathrm{pr}_1^*(K_X))$$

where pr_i are the two projections from $X \times \mathcal{M}_{\mathrm{Bun}_G(X)}$, and

$$\mathrm{ev} : X \times \mathcal{M}_{\mathrm{Bun}_G(X)} \to BG$$

is the evaluation morphism.

2.6. Algebraicity.

Definition 2.35 (Derived algebraic spaces)**.** A derived stack X is 0-*Artin*, or a *derived algebraic space* if its diagonal $X \to X \times X$ is schematic and a monomorphism, and there exists an étale surjection $U \twoheadrightarrow X$ where U is a derived scheme. A morphism $f : X \to Y$ is 0-*Artin*, or *representable*, if for every affine V and every morphism $V \to Y$, the fibre $X \times^{\mathbf{R}}_Y V$ is a derived algebraic space. A 0-Artin morphism $f : X \to Y$ is *flat*, *smooth*, or *surjective* if for every affine V and every morphism $V \to Y$, there exists a derived scheme U and an étale surjection $U \twoheadrightarrow X \times_Y V$ such that the composite $U \twoheadrightarrow X \times_Y V \to V$ has the respective property.

Definition 2.36 (Derived Artin stacks)**.** We define, by induction:

(i) For $n > 0$, a morphism of derived stacks $f : X \to Y$ is $(n-1)$-*Artin* if for every affine V and every morphism $V \to Y$, $X \times^{\mathbf{R}}_Y V$ is $(n-1)$-Artin. A derived stack X is n-*Artin* if its diagonal is $(n-1)$-Artin and there exists a smooth surjection $U \twoheadrightarrow X$ where U is a derived scheme. An $(n-1)$-Artin morphism $f : X \to Y$ is *flat*, *smooth*, or *surjective*, if for any affine V and any morphism $V \to Y$, there exists a derived scheme U and a smooth surjection $U \twoheadrightarrow X \times_Y V$ such that the composite $U \twoheadrightarrow X \times_Y V \to V$ has the respective property.

(ii) A derived stack is *Artin* if it is n-Artin for some n. A morphism of derived stacks is *Artin* if it is n-Artin for some n. A morphism of derived stacks is *flat*, *smooth*, or *surjective*, if it is n-Artin with the respective property for some n.

Definition 2.37 (Derived Deligne–Mumford stacks)**.** A derived 1-Artin stack is *Deligne–Mumford* if it admits an *étale* surjection from a derived scheme. Equivalently, its classical truncation is Deligne–Mumford.

Artin stacks always admit a cotangent complex:

Theorem 2.38. *Let $f : X \to Y$ be an n-Artin morphism of derived stacks.*

(i) *There exists a relative cotangent complex $\mathbf{L}_{X/Y}$ for f.*

(ii) *The cotangent complex $\mathbf{L}_{X/Y}$ is $(-n)$-connective. That is, for every derived scheme U and every smooth morphism $p : U \to X$, the inverse* *image $p^*\mathbf{L}_{X/Y}$ is $(-n)$-connective, i.e.*

$$\mathrm{H}^{-i}(U, \mathbf{L}_{X/Y}) = \pi_i \mathbf{R}\Gamma(U, \mathbf{L}_{X/Y}) = \pi_i \operatorname{Maps}_{\mathrm{D}(U)}(\mathcal{O}_U, p^*\mathbf{L}_{X/Y}) = 0$$

for all $i < -1$. If f is representable by derived algebraic spaces (or derived Deligne–Mumford stacks), then $\mathbf{L}_{X/Y}$ is in fact connective.

The Artin–Lurie representability theorem is a sort of converse:

Theorem 2.39 (Artin–Lurie representability)**.** *Let R be a commutative ring, which we assume is of finite type over a field (or more generally is a G-ring), and X a derived stack over R. Then X is 1-Artin if and only if the following conditions hold:*

(i) *X admits a cotangent complex $\mathbf{L}_X$ (relative to R).*

(ii) *The restriction of X to ordinary R-algebras takes values in 1-groupoids.*

(iii) Almost of finite presentation. *For any $n \geqslant 0$, the functor $X : \mathrm{dCAlg}_R \to \mathrm{Grpd}_\infty$ preserves filtered colimits when restricted to n-truncated algebras.*

(iv) Integrability. *For every complete local noetherian R-algebra A, the canonical map $X(A) \to \varprojlim_n X(A/\mathfrak{m}^n)$ is invertible, where $\mathfrak{m} \subseteq A$ is the maximal ideal.*

(v) Nil-completeness. *For every $A \in \mathrm{dCAlg}_R$, the canonical map $X(A) \to \varprojlim_n X(\tau_{\leqslant n}(A))$ is invertible.*

(vi) Infinitesimal cohesion. *For every cartesian square Q in dCAlg_R of the form*

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow \\ B' & \twoheadrightarrow & B \end{array}$$

such that $A \to B$ and $B' \to B$ are surjective on π_0 with nilpotent kernel, the induced square $X(Q)$ is cartesian.

Example 2.40. Let G be a smooth group scheme over a scheme R and U a derived stack over R with G-action. If U is n-Artin, then so is the quotient stack $[U/G]$. See e.g. [Kha3, Thm. 5.11].

Theorem 2.41. *Let R be a G-ring and X a smooth proper scheme over R. Then the following derived stacks are 1-Artin:*

(i) *The moduli stack $\mathcal{M}_{\mathrm{Vect}(X)}$ of vector bundles over X.*

(ii) *The moduli stack $\mathcal{M}_{\mathrm{Coh}(X)}$ of coherent sheaves on X.*

(iii) *The moduli stack $\mathcal{M}_{\mathrm{Bun}_G(X)}$ of G-bundles over X, for every smooth group scheme G over R.*

Theorem 2.41 can be proven by appealing to Theorem 2.39, where condition (i) is verified using Corollaries 2.32 and 2.34.

2.7. Smoothness.

Example 2.42. If X is a smooth 1-Artin stack over R, then by definition there exists a smooth scheme U and a smooth representable surjection $p : U \twoheadrightarrow X$. We have the exact triangle

$$p^*\mathbf{L}_X \to \mathbf{L}_U \to \mathbf{L}_{U/X}.$$

Since U is smooth, $\mathbf{L}_U \simeq \Omega_U$ is locally free and has Tor-amplitude in $[0,0]$. Since $p : U \twoheadrightarrow X$ is smooth and representable, $\mathbf{L}_{U/X}$ also has Tor-amplitude in $[0,0]$. It follows that the fibre $p^*\mathbf{L}_X$ has Tor-amplitude in $[-1,0]$. Since p is smooth surjective, it follows that $\mathbf{L}_X$ has Tor-amplitude in $[-1,0]$.

More generally we have:

Proposition 2.43. *Let $f : X \to Y$ be a morphism of derived Artin stacks. Then f is smooth if and only if $f_{\mathrm{cl}} : X_{\mathrm{cl}} \to Y_{\mathrm{cl}}$ is locally of finite presentation and $\mathbf{L}_{X/Y}$ is perfect of Tor-amplitude $\leqslant 0$, i.e., if and only if $\mathbf{L}_{X/Y}$ is a perfect complex such that*

$$\pi_i(\mathbf{L}_{X/Y} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}) = 0$$

for all discrete $\mathcal{F} \in \mathrm{D}_{\mathrm{qc}}(X)^{\heartsuit}$ and all $i > 0$.

This suggests the following generalization of smoothness:

Definition 2.44. Let $f : X \to Y$ be a morphism of derived Artin stacks. We say that f is *homotopically smooth* if f_{cl} is locally of finite presentation and $\mathbf{L}_{X/Y}$ is a perfect complex. We say that f is *homotopically n-smooth*, $n \geqslant 0$, if moreover $\mathbf{L}_{X/Y}$ is of Tor-amplitude $\leqslant n$.

Example 2.45. We say that $f : X \to Y$ is *quasi-smooth* if it is homotopically 1-smooth. This admits the following more geometric characterization: there exists a smooth surjection $U \twoheadrightarrow X$ such that $f|_U$ factors via a smooth morphism $Y' \to Y$ and a closed immersion $U \to Y'$ which exhibits U as the derived zero locus of a section s of a vector bundle E over Y'.

$$\begin{array}{ccccc} U & \longrightarrow & Y' & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle s} & & \\ Y' & \xrightarrow{0} & E & & \end{array}$$

See [KRy, Prop. 2.3.14]. In fact, it is possible to generalize this to characterize homotopical n-smoothness by taking into account "shifted" vector bundles $E[-i]$, $0 \leqslant i < n$.

Corollary 2.46. *Let X be a smooth proper scheme over R. The derived stacks $\mathcal{M}_{\mathrm{Vect}(X)}$ and $\mathcal{M}_{\mathrm{Bun}_G(X)}$ (for any smooth group scheme G over R) are homotopically smooth. More precisely, they are homotopically $(n-1)$-smooth if X is of dimension $\leqslant n$.*

Proof. Follows from corollaries 2.32 and 2.34. □

For example, when X is a curve these moduli stacks are smooth (a fortiori flat, hence classical). When X is a surface, they are quasi-smooth, even though their *classical truncations* are generally very singular (not even homotopically smooth). This is a general phenomenon in derived algebraic geometry: derived moduli problems tend to be homotopically smooth. The following fact may be regarded as a conceptual explanation for this phenomenon: it turns out that homotopical smoothness is just the derived analogue of being locally finitely presented.

Theorem 2.47 (Lurie). *A derived stack X over R is homotopically smooth if and only if it is* locally homotopically of finite presentation.

Recall that if X is classical, it is locally of finite presentation if and only if $X : \mathrm{CAlg}_R \to \mathrm{Grpd}$ preserves filtered colimits.

Definition 2.48. A derived stack X is *locally homotopically of finite presentation* (or *locally hfp*) if $X : \mathrm{dCAlg}_R \to \mathrm{Grpd}_\infty$ preserves filtered colimits. (In particular, Theorem 2.39(iii) is automatic in this case.)

More generally, a morphism $X \to Y$ is locally hfp if for every affine $V = \mathrm{Spec}(A)$ and every morphism $\mathrm{Spec}(A) \to Y$, the derived fibre $X \times^{\mathbf{R}}_Y \mathrm{Spec}(A)$ is locally hfp.

Warning 2.49. Sometimes (e.g. in [SAG]), the term *(locally) of finite presentation* is used instead of *(locally) homotopically of finite presentation*. We warn the reader however that the homotopical condition is much stronger than being locally of finite presentation in the sense of classical algebraic geometry. For example, if X and Y are classical noetherian schemes and $i : X \hookrightarrow Y$ is a closed immersion of finite Tor-amplitude (this is automatic say if Y is regular, e.g. smooth over a field), then the following conditions are equivalent (see [Avr, Thm. 1.3]):

(a) i is homotopically of finite presentation, or equivalently homotopically smooth: the relative cotangent complex $\mathbf{L}_{X/Y}$ is a perfect complex;

(b) i is homotopically 1-smooth: the relative cotangent complex $\mathbf{L}_{X/Y}$ is perfect of Tor-amplitude $[0,1]$;

(c) i is a regular (or *lci*) closed immersion.

3. Cohomology of stacks

3.1. Abelian sheaves.

For simplicity, we work over the field $k = \mathbf{C}$ of complex numbers. We denote by Sch_k the category of locally of finite type k-schemes. Given $X \in \mathrm{Sch}_k$ we denote by $\mathrm{D}(X;\mathbf{Z})$ the derived ∞-category of sheaves of abelian groups on X:

$$\mathrm{D}(X;\mathbf{Z}) = \mathrm{Shv}(X(\mathbf{C}); \mathrm{D}(\mathbf{Z}))$$

where $\mathrm{D}(\mathbf{Z})$ is the derived ∞-category of abelian groups (Subsect. 2.3).

Theorem 3.1. *The presheaf $\mathrm{D}^* : \mathrm{Sch}_k^{\mathrm{op}} \to \mathrm{Cat}_\infty$ determined by the assignment*

$$X \mapsto \mathrm{D}(X;\mathbf{Z}), \quad f \mapsto f^* \tag{3.2}$$

satisfies descent for the étale topology. In particular, it satisfies descent for smooth surjections (since they admit étale-local sections).

Construction 3.3. Let AlgStk denote the ∞-category of algebraic stacks locally of finite type over k. By Theorem 3.1, there exists a unique étale sheaf $\mathrm{D}^* : \mathrm{AlgStk}_k^{\mathrm{op}} \to \mathrm{Cat}_\infty$ extending (3.2). More precisely, it is the right Kan extension, given on $X \in \mathrm{AlgStk}_k$ by the formula

$$\mathrm{D}(X) \simeq \varprojlim_{(T,t)} \mathrm{D}(T)$$

where the limit is taken over the category of pairs (T,t) where T is a scheme and $t : T \to X$ is a smooth morphism.

Theorem 3.4 (Six operations)**.** [LZ] *We have the following operations on the ∞-categories $\mathrm{D}(X)$ for $X \in \mathrm{AlgStk}_k$:*

(i) *An adjoint pair of bifunctors*

$$\otimes : \mathrm{D}(X) \times \mathrm{D}(X) \to \mathrm{D}(X),$$
$$\underline{\mathrm{Hom}} : \mathrm{D}(X)^{\mathrm{op}} \times \mathrm{D}(X) \to \mathrm{D}(X)$$

for all $X \in \mathrm{AlgStk}_k$.

(ii) *For every morphism $f : X \to Y$ in AlgStk_k, an adjoint pair*

$$f^* : \mathrm{D}(Y) \to \mathrm{D}(X), \quad f_* : \mathrm{D}(X) \to \mathrm{D}(Y).$$

(iii) *For every morphism $f : X \to Y$ in AlgStk_k, an adjoint pair*

$$f_! : \mathrm{D}(X) \to \mathrm{D}(Y), \quad f^! : \mathrm{D}(Y) \to \mathrm{D}(X).$$

Moreover, they satisfy the following properties:

(i) Base change formula: *For every cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{g} & Y' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{f} & Y \end{array}$$

there is a canonical isomorphism

$$q^* f_! \simeq g_! p^*.$$

(ii) Projection formula: *For every morphism $f : X \to Y$ in AlgStk_k, there is a canonical isomorphism*

$$f_!(-) \otimes (-) \simeq f_!(- \otimes f^*(-)).$$

(iii) Forgetting supports: *If f has proper diagonal, there is a canonical morphism*

$$f_! \to f_*$$

which is invertible when f is proper.

(iv) Étale pull-back: *If f is étale, there is a canonical isomorphism $f^! \simeq f^*$.*

(v) Localization: *If* $X \in \mathrm{Stk}_k$ *and* $i : Z \hookrightarrow X$ *is a closed immersion with complementary open immersion* $j : U \hookrightarrow X$*, then there are canonical exact triangles*

$$j_! j^* \to \mathrm{id} \to i_! i^*$$
$$i_* i^! \to \mathrm{id} \to j_* j^!.$$

Remark 3.5. There is a unique way to extend all the above constructions to *derived* algebraic stacks in such a way that we still have localization triangles: since the inclusion of the classical truncation $i : X_{\mathrm{cl}} \hookrightarrow X$ is a surjective closed immersion, we must have $\mathrm{D}(X) \simeq \mathrm{D}(X_{\mathrm{cl}})$. By base change formulas, all four operations associated with a morphism $f : X \to Y$ must also be identified with the corresponding operations for $f_{\mathrm{cl}} : X_{\mathrm{cl}} \to Y_{\mathrm{cl}}$.

Remark 3.6. Moreover, if we extend $\mathrm{D}(-)$ to *higher* Artin stacks (and thus to all derived Artin stacks) with the same definition, then we still have the six operations in this generality (again, see [LZ]).

3.2. **Co/homology.** Given a (derived) algebraic stack X locally of finite type over k, let $a_X : X \to \mathrm{Spec}(k)$ denote the projection. We define

$$\mathrm{C}^\bullet(X;\mathbf{Z}) := R\Gamma(a_{X,*} a_X^* \mathbf{Z}) \simeq R\Gamma(X;\mathbf{Z}_X),$$
$$\mathrm{C}^{\mathrm{BM}}_\bullet(X;\mathbf{Z}) := R\Gamma(a_{X,*} a_X^! \mathbf{Z}) \simeq R\Gamma(X;\omega_X),$$

where $\mathbf{Z}_X = a_X^* \mathbf{Z}$ and $\omega_X = a_X^! \mathbf{Z}$ denote the constant and dualizing sheaves, respectively. These are the complexes of cochains and Borel–Moore chains on X, respectively. We also write

$$\mathrm{H}^*(X;\mathbf{Z}) := \mathrm{H}^*(\mathrm{C}^\bullet(X;\mathbf{Z})) \simeq \mathrm{H}^*(X;\mathbf{Z}_X),$$
$$\mathrm{H}^{\mathrm{BM}}_*(X;\mathbf{Z}) := \mathrm{H}^{-*}(\mathrm{C}^{\mathrm{BM}}_\bullet(X;\mathbf{Z})) \simeq \mathrm{H}^{-*}(X;\omega_X).$$

Theorem 3.4 yields the following consequences:

Proposition 3.7.

(i) Proper push-forward: *Let* $f : X \to Y$ *be a proper morphism in* AlgStk_k*. Then there is a canonical morphism*

$$f_* : \mathrm{C}^{\mathrm{BM}}_\bullet(X;\mathbf{Z}) \to \mathrm{C}^{\mathrm{BM}}_\bullet(Y;\mathbf{Z}).$$

(ii) Étale pull-back: *Let* $f : X \to Y$ *be an étale morphism in* AlgStk_k*. Then there is a canonical morphism*

$$f^! : \mathrm{C}^{\mathrm{BM}}_\bullet(Y;\mathbf{Z}) \to \mathrm{C}^{\mathrm{BM}}_\bullet(X;\mathbf{Z}).$$

(iii) Localization triangle: *Let* $X \in \mathrm{Stk}_k$ *and* $i : Z \hookrightarrow X$ *a closed immersion with complementary open immersion* $j : U \hookrightarrow X$*. Then there is a canonical exact triangle*

$$\mathrm{C}^{\mathrm{BM}}_\bullet(Z;\mathbf{Z}) \xrightarrow{i_*} \mathrm{C}^{\mathrm{BM}}_\bullet(X;\mathbf{Z}) \xrightarrow{j^!} \mathrm{C}^{\mathrm{BM}}_\bullet(U;\mathbf{Z}).$$

We also have the following consequence of Theorem 3.1:

Corollary 3.8. *On the* ∞*-category* AlgStk_k*, the presheaves*

$$X \mapsto \mathrm{C}^\bullet(X;\mathbf{Z}), \quad f \mapsto f^*$$
$$X \mapsto \mathrm{C}^{\mathrm{BM}}_\bullet(X;\mathbf{Z}), \quad f \mapsto f^!$$

satisfy descent for the étale topology.

Remark 3.9. All these constructions can be done with any reasonable six functor formalism. More precisely, one can work with any topological weave in the sense of [Kha4].[3] Thus for example we can define motivic cohomology and motivic Borel–Moore homology of algebraic stacks satisfying the same properties as above. The discussion throughout this section goes through *mutatis mutandis* in that generality, and the complex $\mathrm{C}^{\mathrm{BM,mot}}_\bullet(-)$ of motivic Borel–Moore chains can be regarded as a "cohomological" and "higher"[4] version of Kresch's Chow groups.

3.3. **Intersection theory.** We are finally in position to see how working with complexes of chains (as objects in the derived ∞-category) rather than their homology groups leads to a streamlined approach to (virtual, stacky) intersection theory. Details of the following constructions can be found in [Kha1].

Definition 3.10. Let $f : X \to Y$ be a homotopically smooth morphism of derived Artin stacks. The *normal bundle* $N_{X/Y}$ is the 1-shifted tangent bundle $T_{X/Y}[1]$; i.e., it is the moduli of sections of the 1-shifted tangent complex $\mathbf{L}^\vee_{X/Y}[1]$. More precisely, it is the derived Artin stack whose functor of points $\mathrm{dSch}^{\mathrm{op}}_X \to \mathrm{Grpd}_\infty$ is given by the assignment

$$(T \xrightarrow{t} X) \mapsto \mathrm{Maps}_{\mathrm{D}_{\mathrm{qc}}(T)}(\mathbf{L}t^*\mathcal{E}, \mathcal{O}_T).$$

Example 3.11. If $f : X \to Y$ is a regular closed immersion between schemes, then the tangent complex $\mathbf{L}^\vee_{X/Y} \simeq \mathcal{N}^\vee_{X/Y}[-1]$ is the shifted normal sheaf, so $N_{X/Y}$ is nothing else than the usual normal bundle.

The following is a generalization of Verdier's deformation to the normal bundle [Ver]:

Definition/Theorem 3.12. *Let* $f : X \to Y$ *be a homotopically smooth morphism of derived Artin stacks. The* normal deformation $D_{X/Y}$ *is the derived mapping stack*

$$D_{X/Y} = \underline{\mathrm{Maps}}_{Y\times\mathbf{A}^1}(Y \times \{0\}, X \times \mathbf{A}^1).$$

(i) *If* X *and* Y *are* n*-Artin, then* $D_{X/Y}$ *is* $(n+1)$*-Artin.*

[3] If the weave does not satisfy étale descent (e.g. the weave of motivic sheaves with integral coefficients), one needs to work with stacks with atlases that admit Nisnevich-local (rather than étale-local) sections. This turns out to be a very mild condition in practice.

[4] in the sense of higher Chow groups or higher algebraic K-theory

(ii) *There is a commutative diagram of cartesian squares*

$$\begin{array}{ccccc} X & \xrightarrow{0} & X \times \mathbf{A}^1 & \longleftarrow & X \times \mathbf{G}_m \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \widehat{f}} & & \downarrow{\scriptstyle f\times \mathrm{id}} \\ N_{X/Y} & \longrightarrow & D_{X/Y} & \longleftarrow & Y \times \mathbf{G}_m \\ \downarrow & & \downarrow & & \downarrow \\ \{0\} & \longrightarrow & \mathbf{A}^1 & \longleftarrow & \mathbf{G}_m. \end{array}$$

See [Kha1, §1.4] and [HKR].

Construction 3.13. Let $f : X \to Y$ be a homotopically smooth morphism of derived algebraic stacks locally of finite type over k. There is a canonical map

$$\mathrm{sp}_{X/Y} : \mathrm{C}^{\mathrm{BM}}_{\bullet}(Y; \mathbf{Z}) \to \mathrm{C}^{\mathrm{BM}}_{\bullet}(N_{X/Y}; \mathbf{Z}) \tag{3.14}$$

defined as the composite

$$\begin{aligned} \mathrm{C}^{\mathrm{BM}}_{\bullet}(Y; \mathbf{Z}) &\xrightarrow{\mathrm{incl}} \mathrm{C}^{\mathrm{BM}}_{\bullet}(Y; \mathbf{Z}) \oplus \mathrm{C}^{\mathrm{BM}}_{\bullet}(Y; \mathbf{Z})(1)[1] \\ &\simeq \mathrm{C}^{\mathrm{BM}}_{\bullet}(Y \times \mathbf{G}_m; \mathbf{Z})[-1] \xrightarrow{\partial} \mathrm{C}^{\mathrm{BM}}_{\bullet}(N_{X/Y}; \mathbf{Z}) \end{aligned}$$

where the splitting comes from the unit section of $\mathbf{G}_m$ and ∂ is the boundary map in the localization triangle

$$\mathrm{C}^{\mathrm{BM}}_{\bullet}(N_{X/Y}; \mathbf{Z}) \to \mathrm{C}^{\mathrm{BM}}_{\bullet}(D_{X/Y}; \mathbf{Z}) \to \mathrm{C}^{\mathrm{BM}}_{\bullet}(Y \times \mathbf{G}_m; \mathbf{Z}) \xrightarrow{\partial}$$

Notation 3.15. For an integer $d \in \mathbf{Z}$, we set $\langle d \rangle := (d)[2d]$, where (d) denotes the Tate twist.

Construction 3.16. Let $f : X \to Y$ be a morphism in AlgStk_k. Suppose $f : X \to Y$ is quasi-smooth, i.e., homotopically 1-smooth (Example 2.45), of relative virtual dimension d. Then $\mathbf{L}_{X/Y}$ is in Tor-amplitude $[-1, 1]$ and $N_{X/Y}$ is a "vector bundle stack". We have the generalized homotopy invariance isomorphism

$$\mathrm{C}^{\mathrm{BM}}_{\bullet}(X; \mathbf{Z}) \simeq \mathrm{C}^{\mathrm{BM}}_{\bullet}(N_{X/Y}; \mathbf{Z})\langle d \rangle.$$

since the projection $N_{X/Y} \to X$ is of relative dimension $-d$. The *quasi-smooth pull-back*, or *virtual pull-back*, is the canonical map

$$f^! : \mathrm{C}^{\mathrm{BM}}_{\bullet}(Y; \mathbf{Z}) \xrightarrow{\mathrm{sp}_{X/Y}} \mathrm{C}^{\mathrm{BM}}_{\bullet}(N_{X/Y}; \mathbf{Z}) \simeq \mathrm{C}^{\mathrm{BM}}_{\bullet}(X; \mathbf{Z})\langle -d \rangle. \tag{3.17}$$

Remark 3.18. Note that, even if X and Y are schemes, the above construction passes through the algebraic stacks $N_{X/Y}$ and $D_{X/Y}$ (which are not schemes unless $f : X \to Y$ is a closed immersion). Similarly, if X and Y are 1-Artin, we need to make use of the extension of $\mathrm{D}(-)$ and the six operations to higher Artin stacks.

Definition 3.19. Let X be a quasi-smooth derived algebraic stack of relative virtual dimension d over $\mathrm{Spec}(k)$. The projection $a_X : X \to \mathrm{Spec}(k)$ gives rise to the pull-back

$$a_X^! : \mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathrm{Spec}(k)) \to \mathrm{C}^{\mathrm{BM}}_{\bullet}(X)\langle -d \rangle$$

and hence to the canonical element

$$[X] \in \mathrm{C}^{\mathrm{BM}}_{\bullet}(X)\langle -d\rangle \quad \rightsquigarrow \quad [X] \in \mathrm{H}^{\mathrm{BM}}_{2d}(X)(-d)$$

called the *virtual fundamental class* of X.

Remark 3.20. The element $[X] \in \mathrm{C}^{\mathrm{BM}}_{\bullet}(X)\langle -d\rangle$ corresponds to a canonical morphism

$$\mathbf{Z}_X\langle d\rangle \to a_X^!(\mathbf{Z})$$

in $\mathrm{D}(X;\mathbf{Z})$. This gives rise to a natural transformation

$$a_X^*(-)\langle d\rangle \to a_X^*(-) \otimes a_X^!(\mathbf{Z}) \xrightarrow{\text{can}} a_X^!(-) \tag{3.21}$$

or by adjunction a trace map $a_{X,!}a_X^*\langle d\rangle \to \mathrm{id}$. In the relative case, where $f : X \to Y$ is a quasi-smooth morphism of relative virtual dimension d, we similarly get a natural transformation

$$\mathrm{tr}_f : f_! f^*\langle d\rangle \to \mathrm{id}. \tag{3.22}$$

Theorem 3.23 (Poincaré duality)**.**

(i) *If $f : X \to Y$ is smooth, then the natural transformation $f^*(-)\langle d\rangle \to f^!(-)$ is invertible. Equivalently,* tr_f *is the counit of an adjunction* $(f_!, f^*\langle d\rangle)$.

(ii) *For any smooth algebraic stack X in* AlgStk_k*, cap product with* $[X]$ *determines a canonical isomorphism*

$$(-) \cap [X] : \mathrm{C}^{\bullet}(X) \to \mathrm{C}^{\mathrm{BM}}_{\bullet}(X)\langle -d\rangle.$$

Proof. If $f : X \to Y$ is smooth, then the diagonal $\Delta : X \to X \times_Y X$ is still quasi-smooth. Thus we have a natural transformation tr_Δ, which gives rise to a unit for the adjunction $(f_!, f^*\langle d\rangle)$. The second statement follows from the first. See [Kha2]. □

Example 3.24. Let $\mathcal{M}_S$ denote the moduli stack $\mathcal{M}_{\mathrm{Coh}(S)}$ (or $\mathcal{M}_{\mathrm{Vect}(S)}$, $\mathcal{M}_{\mathrm{Bun}_G(S)}$) for S an algebraic surface. Then since $\mathcal{M}_S$ is quasi-smooth (Corollary 2.46), we have constructed a (virtual) fundamental class $[\mathcal{M}_S] \in \mathrm{H}^{\mathrm{BM}}_*(\mathcal{M}_S)$. Note that the traditional method [BF] does not apply here since $\mathcal{M}_S$ is far from being Deligne–Mumford.

In this framework it is easy to prove the following formula for intersection products. If X is a smooth k-scheme, the cap product in cohomology gives rise by Poincaré duality to an intersection product

$$\mathrm{C}^{\mathrm{BM}}_{\bullet}(X)\langle -p\rangle \otimes \mathrm{C}^{\mathrm{BM}}_{\bullet}(X)\langle -q\rangle \to \mathrm{C}^{\mathrm{BM}}_{\bullet}(X)\langle -p-q+d\rangle.$$

If Y is quasi-smooth of virtual dimension d and proper over X, the virtual fundamental class gives rise to a class in $\mathrm{C}^{\mathrm{BM}}_{\bullet}(X)\langle -d\rangle$ by proper push-forward.

Theorem 3.25 (Non-transverse Bézout formula)**.** *Let Y and Z be smooth or lci closed subvarieties of X, of dimension p and q respectively. Then there is a canonical homotopy*

$$[Y] \cdot [Z] \simeq [Y \overset{\mathbf{R}}{\underset{X}{\times}} Z]$$

in $\mathrm{C}^{\mathrm{BM}}_\bullet(X)\langle -p-q+d\rangle$.

Note that while the left-hand side consists of usual cycle classes, the right-hand side is genuinely virtual unless the intersection is transverse (that is to say, unless the derived intersection $Y \times^{\mathbf{R}}_X Z$ reduces to the classical scheme-theoretic intersection).

3.4. **Quotient stacks.**

Definition 3.26. Let G be a linear algebraic group over the base field k. Let $X \in \mathrm{AlgStk}_k$ be an algebraic stack with G-action. The complex of *equivariant Borel–Moore chains* is defined by

$$\mathrm{C}^{\mathrm{BM},G}_\bullet(X) \coloneqq R\Gamma([X/G], f^!(\mathbf{Z}_{BG})) \simeq \mathrm{C}^{\mathrm{BM}}_\bullet([X/G];\mathbf{Z})\langle g\rangle$$

where $f : [X/G] \to BG$ is the projection of the quotient stack to the classifying stack, $g = \dim(G)$, and the isomorphism is Poincaré duality for BG.

The following two statements, proven in [KRa], show that this construction can be described by (algebraic approximations to) the Borel construction.

Choose a filtered system $(V_\alpha)_\alpha$ of G-representations where the transition maps $V_\alpha \hookrightarrow V_\beta$ are monomorphisms. Let $W_\alpha \subseteq V_\alpha$ be G-invariant closed subschemes such that:

(a) G acts freely on $U_\alpha \coloneqq V_\alpha \smallsetminus W_\alpha$,

(b) $U_\alpha \subseteq U_{\alpha+1}$ for all α,

(c) We have $\mathrm{codim}_{V_\alpha}(W_\alpha) \to \infty$ as $n \to \infty$.

Let U_∞ denote the ind-algebraic space $\{U_\alpha\}_\alpha$. For example, for $G = \mathbf{G}_m$ the obvious choices give $[U_\infty/G] = \mathbf{P}^\infty_k$.

Theorem 3.27. *There is a canonical isomorphism*

$$\mathrm{C}^{\mathrm{BM},G}_\bullet(X) \simeq \mathrm{C}^{\mathrm{BM}}_\bullet(X \overset{G}{\times} U_\infty)\langle -\dim(U_\infty/G)\rangle \coloneqq \varprojlim_\alpha \mathrm{C}^{\mathrm{BM}}_\bullet(X \overset{G}{\times} U_\alpha)\langle -d_\alpha\rangle$$

where $X \times^G U_\alpha \coloneqq [(X \times U_\alpha)/G]$ *is the quotient by the (free) diagonal action and* $d_\alpha = \dim(U_\alpha/G)$.

Theorem 3.28. *There is a cartesian square of* ∞*-categories*

$$\begin{array}{ccc}
\mathrm{D}([X/G]) & \hookrightarrow & \mathrm{D}(X \times^G U_\infty) \\
\downarrow & & \downarrow \\
\mathrm{D}(X) & \hookrightarrow & \mathrm{D}(X \times U_\infty)
\end{array}$$

where every arrow is $*$*-pullback, and the horizontal arrows are fully faithful.*

Informally speaking, this means that a sheaf on $[X/G]$ amounts to the data of a sheaf $\mathcal{F}$ on X, a sheaf $\mathcal{G}$ on $X \times^G U_\infty$, and an isomorphism $\mathcal{F}|_{X\times U_\infty} \simeq \mathcal{G}|_{X\times U_\infty}$.

3.5. **Concentration and localization.** Let $X \in \mathrm{AlgStk}_k$ and let $i : Z \hookrightarrow X$ be a closed immersion. Let Σ be a set of line bundles on X.

Definition 3.29 (Concentration)**.** We say that i satisfies *concentration* with respect to Σ if the induced map

$$i_* : \mathrm{C}^{\mathrm{BM}}_\bullet(Z;\mathbf{Z})[c_1(\Sigma)^{-1}] \to \mathrm{C}^{\mathrm{BM}}_\bullet(X;\mathbf{Z})[c_1(\Sigma)^{-1}] \tag{3.30}$$

is invertible.

The following was proven in [AKLPR]:

Theorem 3.31. *Assume that X has affine stabilizers. Suppose $Z \subseteq X$ is a closed substack such that for every point $x \in X \smallsetminus Z$ there exists a line bundle $L \in \Sigma$ whose restriction along $B\,\mathrm{Aut}(x) \hookrightarrow X$ is trivial. Then $Z \hookrightarrow X$ satisfies concentration, i.e., (3.30) is invertible.*

Corollary 3.32. *Let T be a split algebraic torus acting on a Deligne–Mumford stack $X \in \mathrm{AlgStk}_k$. Let Z be the closed substack of fixed points*[5]*. Then $i : Z \hookrightarrow X$ satisfies concentration with respect to the set Σ all nontrivial characters of BT (pulled back to $[X/T]$): in particular, we have a canonical isomorphism*

$$i_* : \mathrm{C}^{\mathrm{BM},T}_\bullet(Z;\mathbf{Z})[c_1(\Sigma^{-1})] \to \mathrm{C}^{\mathrm{BM},T}_\bullet(X;\mathbf{Z})[c_1(\Sigma)^{-1}].$$

The localization triangle (Proposition 3.7) gives a very useful way to prove results of this form, since it reduces the problem to Σ-acyclicity of Borel–Moore chains on the complement $X \smallsetminus Z$.

From this one can derive:

Corollary 3.33 (Virtual localization)**.** *Let T be a split algebraic torus acting on a Deligne–Mumford stack $X \in \mathrm{AlgStk}_k$. Assume X is quasi-smooth and let Z be the fixed locus as in Corollary 3.32. Then we have a canonical homotopy*

$$[X] \simeq i_*([Z] \cap e(N_{Z/X})^{-1})$$

in $\mathrm{C}^{\mathrm{BM},T}_\bullet(X)[c_1(\Sigma^{-1})]$.

When X is smooth this is the Atiyah–Bott localization formula. In the quasi-smooth case it is the virtual localization formula of Graber–Pandharipande [GP]. Unlike *op. cit.* we do not need to assume X admits a global embedding into an ambient smooth stack, or that the cotangent complex $\mathbf{L}_{Z/X}$ admits a global resolution by vector bundles. Again, these improvements are possible because we work at the level of Borel–Moore chains as objects of the derived ∞-category $\mathrm{D}(\mathbf{Z})$.

[5] Since X is a stack, the appropriate definition of Z here is subtle. Briefly, Z is the homotopy fixed point stack with respect to an appropriate reparametrization of the torus action. See [AKLPR, Cor. 3.7].

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