Lecture 9

Pro-cdh excision in K-theory

Let $\mathrm{Z} \hookrightarrow \mathrm{X}$ be a closed immersion of classical schemes. If $\mathrm{X} = \mathrm{Spec}(\mathrm{R})$ is affine, we can choose generators $f_1, \ldots, f_r$ for the ideal of definition and form the derived regular immersion

$$\tilde{\mathrm{Z}} = \mathrm{Spec}(\mathrm{R}/\!\!/(f_1, \ldots, f_r)) \hookrightarrow \mathrm{X}.$$

We can then form the derived blow-up $\mathrm{Bl}_{\tilde{\mathrm{Z}}/\mathrm{X}} \to \mathrm{X}$, and, as we have seen in the last few lectures, there is a cartesian square of K-theory spectra

$$\begin{array}{ccc} \mathrm{K}(\mathrm{X}) & \longrightarrow & \mathrm{K}(\tilde{\mathrm{Z}}) \\ \downarrow & & \downarrow \\ \mathrm{K}(\mathrm{Bl}_{\tilde{\mathrm{Z}}/\mathrm{X}}) & \longrightarrow & \mathrm{K}(\mathbf{P}_{\mathrm{Z}}(\mathcal{N}_{\tilde{\mathrm{Z}}/\mathrm{X}})). \end{array}$$

The goal of this lecture is to explain how we can derive from this a statement involving only classical schemes and their classical blow-ups.

1. The pro cdh excision theorem.

Definition 1.1. *An* abstract blow-up square *is a cartesian square of classical schemes*

$$\begin{array}{ccc} \mathrm{E} & \hookrightarrow & \mathrm{Y} \\ \downarrow & & \downarrow p \\ \mathrm{Z} & \xhookrightarrow{i} & \mathrm{X} \end{array}$$

where $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ is a closed immersion, and $p : \mathrm{Y} \to \mathrm{X}$ is a proper morphism that induces an isomorphism $p : \mathrm{Y} - \mathrm{E} \xrightarrow{\sim} \mathrm{X} - \mathrm{Z}$.

Theorem 1.2. *Suppose we have an abstract blow-up square*

$$\begin{array}{ccc} \mathrm{E} & \hookrightarrow & \mathrm{Y} \\ \downarrow & & \downarrow \\ \mathrm{Z} & \hookrightarrow & \mathrm{X} \end{array}$$

of noetherian classical schemes. Let $\mathrm{Z}^{(n)}$, resp. $\mathrm{E}^{(n)}$, denote the $(n-1)$-st infinitesimal neighbourhood of Z in X, resp. of E in Y, for $n > 0$. Then the induced square of pro-spectra

$$\begin{array}{ccc} \{\mathrm{K}(\mathrm{X})\} & \longrightarrow & \{\mathrm{K}(\mathrm{Z}^{(n)})\}_{n>0} \\ \downarrow & & \downarrow \\ \{\mathrm{K}(\mathrm{Y})\} & \longrightarrow & \{\mathrm{K}(\mathrm{E}^{(n)})\}_{n>0} \end{array}$$

is quasi-cartesian, i.e. the morphism

$$\{\mathrm{K}(\mathrm{X})\} \to \{\mathrm{K}(\mathrm{Z}^{(n)}) \times_{\mathrm{K}(\mathrm{E}^{(n)})} \mathrm{K}(\mathrm{Y})\}_n$$

is a quasi-isomorphism of pro-spectra.

1.3. Let us only mention in passing that Weibel's conjecture, which asserts that for a noetherian scheme $\mathrm{K}(\mathrm{X})$ of dimension d, the spectrum $\mathrm{K}(\mathrm{X})$ is $(-d)$-connective, is an immediate consequence of pro-cdh excision together with the following theorem of Kerz–Strunk:

Theorem 1.4. *Let X be a reduced affine noetherian scheme. Then for any negative K-theory class $x \in \mathrm{K}_{-i}(\mathrm{X})$ $(i > 0)$, there exists a cdh cover $f : \mathrm{Y} \to \mathrm{X}$ such that the inverse image $f^*(x) \in \mathrm{K}_{-i}(\mathrm{Y})$ vanishes.*

1.5. It will be convenient to adopt the following notation: for a morphism $\mathrm{Y} \to \mathrm{X}$, write $\mathrm{K}(\mathrm{X},\mathrm{Y})$ for the *relative K-theory spectrum*, the homotopy fibre

$$\mathrm{K}(\mathrm{X},\mathrm{Y}) = \mathrm{Fib}(\mathrm{K}(\mathrm{X}) \to \mathrm{K}(\mathrm{Y})).$$

For a morphism of simplicial commutative rings $\mathrm{A} \to \mathrm{B}$ we write $\mathrm{K}(\mathrm{A},\mathrm{B}) := \mathrm{K}(\mathrm{Spec}(\mathrm{A}),\mathrm{Spec}(\mathrm{B}))$.

It is easy to see that the statement of the theorem can then be reformulated as the assertion that the canonical map

$$\{\mathrm{K}(\mathrm{X},\mathrm{Z}^{(n)})\}_{n>0} \to \{\mathrm{K}(\mathrm{Y},\mathrm{E}^{(n)})\}_{n>0}$$

is a quasi-isomorphism.

Remark 1.6. Warning: for a closed immersion i, there is generally no identification $\mathrm{K}(\mathrm{X},\mathrm{Z}) = \mathrm{K}(\mathrm{Perf}(\mathrm{X},\mathrm{Z}))$, where $\mathrm{Perf}(\mathrm{X},\mathrm{Z}) = \mathrm{Ker}(\mathrm{Perf}(\mathrm{X}) \to \mathrm{Perf}(\mathrm{Z}))$.

2. Stategy of proof.

2.1. Here we will restrict our attention to the case of actual blow-up squares, i.e. $\mathrm{Y} = \mathrm{B}^{\mathrm{cl}} := \mathrm{Bl}^{\mathrm{cl}}_{\mathrm{Z}/\mathrm{X}}$. In general one can reduce to this case by a certain argument involving Raynaud–Gruson's technique of "platification par éclatement".

Using Zariski descent in K-theory (Lect. 4) we can immediately reduce to the case where $\mathrm{X} = \mathrm{Spec}(\mathrm{R})$ is affine (with R noetherian). Let $f_1, \ldots, f_r$ be (an arbitrary choice of) generators of the ideal defining Z; these determine a derived thickening $\tilde{\mathrm{Z}} = \mathrm{Spec}(\mathrm{R}/\!\!/(f_i)_i)$. Let $\mathrm{B}^{\mathrm{der}} = \mathrm{Bl}_{\tilde{\mathrm{Z}}/\mathrm{X}}$ denote the derived blow-up and let $\mathrm{D} \hookrightarrow \mathrm{B}^{\mathrm{der}}$ denote the virtual exceptional divisor. Recall from Lect. 7 that in this case the derived scheme $\mathrm{B}^{\mathrm{der}}$ is the derived base change $\mathrm{Bl}_{\{0\}/\mathbf{A}^r} \times_{\mathbf{A}^r} \mathrm{X}$; similarly D is the derived base change of the exceptional divisor in $\mathrm{Bl}_{\{0\}/\mathbf{A}^r}$.

For each $n > 0$ set $\tilde{\mathrm{Z}}^{(n)} = \mathrm{Spec}(\mathrm{R}/\!\!/(f_i^n)_i)$, let $(\mathrm{B}^{\mathrm{der}})^{(n)} = \mathrm{Bl}_{\tilde{\mathrm{Z}}^{(n)}/\mathrm{X}}$, and $\mathrm{D}^{(n)} \hookrightarrow (\mathrm{B}^{\mathrm{der}})^{(n)}$ the virtual exceptional divisor.

2.2. For each $n > 0$ we have morphisms of squares

$$\begin{array}{ccc} \mathrm{E}^{(n)} & \hookrightarrow & \mathrm{B}^{\mathrm{cl}} \\ \downarrow & & \downarrow \\ \mathrm{Z}^{(n)} & \hookrightarrow & \mathrm{X} \end{array} \quad \to \quad \begin{array}{ccc} \mathrm{D}^{(n)}_{\mathrm{cl}} & \hookrightarrow & (\mathrm{B}^{\mathrm{der}})_{\mathrm{cl}} \\ \downarrow & & \downarrow \\ \mathrm{Z}^{(n)} & \hookrightarrow & \mathrm{X} \end{array} \quad \to \quad \begin{array}{ccc} \mathrm{D}^{(n)} & \hookrightarrow & \mathrm{B}^{\mathrm{der}} \\ \downarrow & & \downarrow \\ \tilde{\mathrm{Z}}^{(n)} & \hookrightarrow & \mathrm{X} \end{array}$$

going from:

- the classical blow-up B^{cl}, to
- the underlying classical scheme of the derived blow-up $(\mathrm{B}^{\mathrm{der}})_{\mathrm{cl}}$, to
- the derived blow-up $\mathrm{B}^{\mathrm{der}}$.

We can express the composite arrow as a commutative diagram

$$\begin{array}{ccc} \{(\mathrm{B}^{\mathrm{cl}},\mathrm{E}^{(n)})\}_n & \longrightarrow & \{(\mathrm{B}^{\mathrm{der}},\mathrm{D}^{(n)})\}_n \\ \downarrow & & \downarrow \\ \{(\mathrm{X},\mathrm{Z}^{(n)})\}_n & \longrightarrow & \{(\mathrm{X},\tilde{\mathrm{Z}}^{(n)})\}_n. \end{array}$$

In order to show that the left-hand arrow induces a quasi-isomorphism on K-theory pro-spectra, it will suffice to show this for the other three arrows in this square.

2.3. For the lower horizontal arrow, this follows from the quasi-isomorphism

$$\{\mathrm{K}(\tilde{\mathrm{Z}}^{(n)})\}_n \xrightarrow{\sim} \{\mathrm{K}(\mathrm{Z}^{(n)})\}_n$$

demonstrated in Lect. 8.

2.4. For the right-hand arrow the claim is a variation on the derived blow-up formula: it is an isomorphism in level $n = 0$, and it turns out also to be a quasi-isomorphism as n varies.

2.5. For time reasons we will focus on the upper horizontal arrow, relating the classical blow-up to the derived blow-up, which is the most involved part of the proof.

Recall that the derived blow-up square

$$\begin{array}{ccc} \mathrm{D}^{(n)} & \hookrightarrow & \mathrm{B}^{\mathrm{der}} \\ \downarrow & & \downarrow \\ \tilde{\mathrm{Z}}^{(n)} & \hookrightarrow & \mathrm{X} \end{array}$$

is never cartesian. The canonical morphism

$$\delta^{(n)} : \mathrm{D}^{(n)} \to \mathrm{B}^{\mathrm{der}} \underset{\mathrm{X}}{\times} \tilde{\mathrm{Z}}^{(n)} := \mathrm{W}^{(n)}$$

is nevertheless a nil-immersion, i.e. it induces a levelwise isomorphism

$$\delta : \{\mathrm{D}^{(n)}_{\mathrm{cl}}\}_n \to \{\mathrm{W}^{(n)}_{\mathrm{cl}}\}_n.$$

Thus the upper horizontal arrow factors through morphisms

$$\{(\mathrm{B}^{\mathrm{cl}}, \mathrm{E}^{(n)})\}_n \xrightarrow{\alpha_1} \{((\mathrm{B}^{\mathrm{der}})_{\mathrm{cl}}, \mathrm{D}^{(n)}_{\mathrm{cl}})\}_n = \{((\mathrm{B}^{\mathrm{der}})_{\mathrm{cl}}, \mathrm{W}^{(n)}_{\mathrm{cl}})\}_n \xrightarrow{\alpha_2} \{(\mathrm{B}^{\mathrm{der}}, \mathrm{W}^{(n)})\}_n \xrightarrow{\alpha_3} \{(\mathrm{B}^{\mathrm{der}}, \mathrm{D}^{(n)})\}_n.$$

We will show that each of these induces an isomorphism on K-theory pro-spectra.

3. Pro Milnor excision. Before proceeding to the proof we now state a couple pro versions of Milnor excision, which are the main tools we will use. We will come back to their proofs next lecture.

Theorem 3.1. *Let* $\mathrm{A} \to \mathrm{B}$ *be a homomorphism of (discrete) noetherian commutative rings, and* $\mathrm{I} \subset \mathrm{A}$ *an ideal which maps isomorphically onto an ideal* $\mathrm{J} \subset \mathrm{B}$. *Then the morphism of pro-spectra*

$$\{\mathrm{K}(\mathrm{A}, \mathrm{A}/\mathrm{I}^n)\}_{n>0} \to \{\mathrm{K}(\mathrm{B}, \mathrm{B}/\mathrm{J}^n)\}_{n>0}$$

is a quasi-isomorphism.

Theorem 3.2. *Let* R *be a noetherian simplicial commutative ring and* $(f_1, \ldots, f_r)$ *a sequence of elements. Suppose that the pro-abelian group* $\{(f_i^n)_i(\pi_k\mathrm{R})\}_n$ *vanishes for each* $k > 0$. *Then the morphism of pro-spectra*

$$\{\mathrm{K}(\mathrm{R}, \mathrm{R}/\!\!/(f_i^n)_i\}_{n>0} \to \{\mathrm{K}(\pi_0\mathrm{R}, \pi_0\mathrm{R}/(f_i^n)_i\}_{n>0}$$

is a quasi-isomorphism.

Remark 3.3. The condition in Theorem 3.2 holds if we assume that the open complement of the closed derived subscheme $\mathrm{Spec}(\mathrm{R}/\!\!/(f_i)_i) \hookrightarrow \mathrm{Spec}(\mathrm{R})$ is a classical scheme. Indeed the latter condition amounts to saying that the localizations $\mathrm{R}[f_i^{-1}]$ are all discrete, or equivalently that $f_i^m(\pi_k\mathrm{R}) = 0$ for some m and each i.

4. Step 1. We begin by considering the morphism $\alpha_1 : (\mathrm{B}^{\mathrm{cl}}, \mathrm{E}^{(n)}) \to ((\mathrm{B}^{\mathrm{der}})_{\mathrm{cl}}, \mathrm{D}_{\mathrm{cl}}^{(n)})$ which is induced by the canonical inclusion

$$\mathrm{B}^{\mathrm{cl}} \to (\mathrm{B}^{\mathrm{der}})_{\mathrm{cl}}$$

of the classical blow-up into the classical scheme underlying the derived blow-up.

Claim 4.1. *The map of pro-spectra*

$$\{\mathrm{K}((\mathrm{B}^{\mathrm{der}})_{\mathrm{cl}}, \mathrm{D}_{\mathrm{cl}}^{(n)})\}_n \to \{\mathrm{K}(\mathrm{B}^{\mathrm{cl}}, \mathrm{E}^{(n)})\}_n \tag{4.1}$$

is a quasi-isomorphism.

4.2. To prove this we will use the following consequence of pro-Milnor excision:

Proposition 4.3 (Pro closed gluing)**.** *Let*

$$\begin{array}{ccc} \mathrm{E} & \hookrightarrow & \mathrm{Y} \\ \downarrow & & \downarrow \\ \mathrm{Z} & \hookrightarrow & \mathrm{X} \end{array}$$

be an abstract blow-up square of noetherian classical schemes. Suppose that $\mathrm{Y} \to \mathrm{X}$ *is a closed immersion. Then* K*-theory satisfies pro excision for this square.*

Proof. Again we can assume $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$ is affine by Zariski descent. Let $\mathrm{Z} = \mathrm{Spec}(\mathrm{A}/\mathrm{I})$ and $\mathrm{Y} = \mathrm{Spec}(\mathrm{A}/\mathrm{J})$. The condition that $\mathrm{Y} \to \mathrm{X}$ is an isomorphism away from Z implies that the homomorphism $\mathrm{A} \to \mathrm{A}/\mathrm{J}$ induces an isomorphism $\mathrm{A}_f \to (\mathrm{A}/\mathrm{J})_f = \mathrm{A}_f/\mathrm{J}_f$ for each element $f \in \mathrm{I}$. This means that there exists some $s > 0$ such that $f^s \cdot \mathrm{J} = 0$ for some $s > 0$, for each $f \in \mathrm{I}$; in particular we find $\mathrm{I}^s \cdot \mathrm{J} = 0$ for sufficiently large s. By the Artin–Rees lemma there exists an integer $t > 0$ such that $\mathrm{I}^{i+t} \cap \mathrm{J} = \mathrm{I}^i(\mathrm{I}^t \cap \mathrm{J})$ for all $i \geqslant 0$. Thus taking $i \geqslant s$, we conclude that $\mathrm{I}^k \cap \mathrm{J} = 0$ for some $k \gg 0$. In other words, the homomorphism $\mathrm{A} \to \mathrm{A}/\mathrm{J}$ sends the ideal I^k isomorphically onto an ideal of A/J. Therefore, by pro Milnor excision (Theorem 3.1), we have a quasi-isomorphism

$$\{\mathrm{K}(\mathrm{X}, \mathrm{Z}^{(kn)})\}_n \xrightarrow{\sim} \{\mathrm{K}(\mathrm{Y}, \mathrm{E}^{(kn)})\}_n,$$

whence the claim. □

4.4. To apply this in our situation, we first note:

Claim 4.5. *The morphism of classical* X*-schemes* $\mathrm{B}^{\mathrm{cl}} \to (\mathrm{B}^{\mathrm{der}})_{\mathrm{cl}}$ *is a closed immersion which is an isomorphism over the complement* $\mathrm{X} - \mathrm{Z}$.

Proof. The second part of the claim is obvious from the construction.

Over the closed subscheme Z, the fibre of B^{cl} is the (classical) exceptional divisor E, which is isomorphic to the projectivized normal cone

$$\mathrm{E} = \mathrm{Proj}_{\mathrm{Z}}(\mathcal{C}_{\mathrm{Z}/\mathrm{X}}),$$

where $\mathcal{C}_{\mathrm{Z}/\mathrm{X}}$ is the (discrete) graded quasi-coherent $\mathcal{O}_{\mathrm{Z}}$-algebra $\bigoplus_{k\geqslant 0} \mathcal{I}^k/\mathcal{I}^{k+1}$. The fibre of $(\mathrm{B}^{\mathrm{der}})_{\mathrm{cl}}$ over Z is isomorphic to the "projectivized virtual normal bundle"

$$(\mathbf{P}_{\tilde{\mathrm{Z}}}(\mathcal{N}_{\tilde{\mathrm{Z}}/\mathrm{X}}))_{\mathrm{cl}} = \mathbf{P}_{\mathrm{Z}}(i_0^* \mathcal{N}_{\tilde{\mathrm{Z}}/\mathrm{X}})$$

where $i_0 : \mathrm{Z} \hookrightarrow \tilde{\mathrm{Z}}$ is the inclusion. Thus the claim follows from the next lemma. □

Lemma 4.6. *Let* X *be a classical scheme,* $i : \tilde{\mathrm{Z}} \hookrightarrow \mathrm{X}$ *a regular closed immersion, and* $i_0 : \mathrm{Z} \hookrightarrow \tilde{\mathrm{Z}}$ *the inclusion of the underlying classical scheme. Then the canonical morphism of quasi-coherent* $\mathcal{O}_{\mathrm{Z}}$*-algebras*

$$\mathrm{Sym}_{\mathcal{O}_{\mathrm{Z}}}(i_0^* \mathcal{N}_{\tilde{\mathrm{Z}}/\mathrm{X}}) \to \mathcal{C}_{\mathrm{Z}/\mathrm{X}}$$

is surjective.

Proof. The morphism in question factors through the canonical surjection

$$\mathrm{Sym}_{\mathcal{O}_Z}(\mathcal{I}/\mathcal{I}^2) \twoheadrightarrow \mathcal{C}_{Z/X},$$

so it suffices to show that the morphism of quasi-coherent $\mathcal{O}_Z$-modules

$$i_0^*\mathcal{N}_{\tilde{Z}/X} \to \mathcal{I}/\mathcal{I}^2$$

is surjective. The claim is local and is not difficult to check using the "connectivity lemma" for the cotangent complex (Lect. 6, Lem. 4.13). □

4.7. In view of the above we get an abstract blow-up square

$$\begin{array}{ccc} E & \hookrightarrow & B^{cl} \\ \downarrow & & \downarrow \\ D_{cl} & \hookrightarrow & (B^{der})_{cl}. \end{array}$$

By pro closed gluing (Proposition 4.3) we conclude the proof of Claim 4.1.

5. Step 2. We next consider the canonical morphism

$$\alpha_2 : ((B^{der})_{cl}, W^{(n)}_{cl}) \to (B^{der}, W^{(n)}).$$

Claim 5.1. *The induced map of pro-spectra*

$$(\alpha_2)^* : \{K(B^{der}, W^{(n)})\}_n \to \{K((B^{der})_{cl}, W^{(n)}_{cl})\}_n$$

is a quasi-isomorphism.

Proof. This follows from Zariski descent and pro Milnor excision (Theorem 3.2), since the open complement $B^{der} - W$ is isomorphic to the classical scheme $X - Z$. □

6. Step 3. Finally we consider the morphism

$$\alpha_3 : (B^{der}, D^{(n)}) \to (B^{der}, W^{(n)}).$$

Claim 6.1. *The map of pro-spectra* $(\alpha_3)^* : \{K(W^{(n)})\}_n \to \{K(D^{(n)})\}_n$ *is a quasi-isomorphism. In particular, the map*

$$\{K(B^{der}, W^{(n)})\}_n \to \{K(B^{der}, D^{(n)})\}_n \tag{6.1}$$

is a quasi-isomorphism.

Proof. We claim that the morphism $\{D^{(n)}\}_n \to \{W^{(n)}\}_n$ is locally a quasi-isomorphism of pro-simplicial rings. The desired conclusion will follow from this in view of the fact that K-theory prserves quasi-isomorphisms of pro-simplicial rings, and satisfies Zariski descent.

To prove this, recall that in our situation the closed immersion $\tilde{Z} \hookrightarrow X$ is a derived base change

$$\begin{array}{ccc} \tilde{Z} & \hookrightarrow & X \\ \downarrow & & \downarrow f \\ \{0\} & \hookrightarrow & \mathbf{A}^r. \end{array}$$

Therefore the derived blow-up $\mathrm{Bl}_{\tilde{Z}/X}$ is a derived base change of $\mathrm{Bl}_{\{0\}/\mathbf{A}^r}$ along $f : X \to \mathbf{A}^r$ (the morphism determined by the sections $f_1, \ldots, f_r$). In particular the standard affine charts U_i of $\mathrm{Bl}_{\{0\}/\mathbf{A}^r}$ induce affine charts V_i of $\mathrm{Bl}_{\tilde{Z}/X}$, and it will suffice to show that

$$\{D^{(n)} \cap V_i\}_n \to \{W^{(n)} \cap V_i\}_n$$

corresponds to a quasi-isomorphism of pro-simplicial rings (more precisely, for any intersection of V_i's). Furthermore, by base change we can also replace $\tilde{\mathrm{Z}} \hookrightarrow \mathrm{X}$ by $\{0\} \hookrightarrow \mathbf{A}^r$, i.e. we need to consider

$$\{\mathrm{E}_i^{(n)}\}_n \to \{\mathrm{U}_i \underset{\mathbf{A}^r}{\times} \{0\}^{(n)}\}_n$$

for each i. Here $\mathrm{U}_i = \mathrm{Spec}(\mathrm{A}_i)$, $\mathrm{A}_i = \mathbf{Z}[x_1/x_i, \ldots, x_n/x_i, x_i]$, form the standard affine cover of $\mathrm{Bl}_{\{0\}/\mathbf{A}^n}$, and $\mathrm{E}_i = \mathrm{Spec}(\mathrm{A}_i/x_i)$ form the standard affine cover of the exceptional divisor. Thus we are looking at the morphism of pro-simplicial rings

$$\{\mathrm{A}_i /\!\!/ (x_1^n, \ldots, x_r^n)\}_n \to \{\mathrm{A}_i/(x_i^n)\}_n$$

which is a 0-truncation since the ideals $(x_1^n, \ldots, x_r^n) = (x_i^n)$ are equal in the ring A_i. Hence the claim follows from Prop. 5.1 from Lect. 8. □

References.

[1] M. Kerz, F. Strunk, *On the vanishing of negative homotopy K-theory.*
[2] M. Kerz, F. Strunk, G. Tamme, *Algebraic K-theory and descent for blow-ups.*
[3] Matthew Morrow, *Pro cdh-descent for cyclic homology and* K*-theory.*