Lecture 2

Descent for quasi-coherent sheaves

In this lecture we will continue our study of quasi-coherent sheaves by proving a descent theorem and looking at some of its basic consequences.

1. Fpqc descent.

1.1. We begin with the derived analogue of the "mother of all descent theorems", which is Grothendieck's faithfully flat descent:

Theorem 1.2. *The presheaf of ∞-categories on the site of affine derived schemes*

$$\mathrm{S} \mapsto \mathrm{Qcoh}(\mathrm{S})$$

satisfies fpqc descent.

We can replace $\mathrm{Qcoh}(-)$ by $\mathrm{Qcoh}(-)_{\geqslant 0}$: one recovers $\mathrm{Qcoh}(-)$ by stabilizing, which commutes with limits when the transition functors in the diagram are left-exact. Let $(f_\alpha : \mathrm{S}_\alpha \to \mathrm{S})_\alpha$ be an fpqc-covering family and let $f : \tilde{\mathrm{S}} \to \mathrm{S}$ where $\tilde{\mathrm{S}} = \coprod_\alpha \mathrm{S}_\alpha$. We want to show that the canonical functor

$$\mathrm{Qcoh}(\mathrm{S})_{\geqslant 0} \to \mathrm{Tot}(\mathrm{Qcoh}(\check{\mathrm{C}}(\tilde{\mathrm{S}}/\mathrm{S})_\bullet)_{\geqslant 0})$$

is an equivalence, where we have adopted the notation $\mathrm{Tot}(\mathrm{A}^\bullet) := \varprojlim_{n\in\mathbf{\Delta}} \mathrm{A}^n$ for the totalization or limit of a cosimplicial diagram $\mathrm{A}^\bullet$. This totalization can be identified with the ∞-category of co-algebras in $\mathrm{Qcoh}(\tilde{\mathrm{S}})_{\geqslant 0}$ over the comonad associated to the adjunction $f^* : \mathrm{Qcoh}(\mathrm{S})_{\geqslant 0} \rightleftarrows \mathrm{Qcoh}(\tilde{\mathrm{S}})_{\geqslant 0} : f_*$ [3, Lem. D.3.5.7]. Thus it suffices to show that this adjunction is comonadic, for which we can apply the Barr–Beck–Lurie theorem to check two conditions:

(i) The functor f^* is conservative.

(ii) The functor f_*f^* preserves limits of cosimplicial diagrams that admit a splitting after applying f^*.

The first holds by definition of faithfully flat morphism. The second is a more involved Bousfield–Kan type argument which we briefly sketch here (see [3, Prop. D.6.4.6] for details). Let $\mathcal{G}^\bullet$ be a cosimplicial diagram in $\mathrm{Qcoh}(\mathrm{S})_{\geqslant 0}$ which is f^*-split; the claim is that the canonical map $f_*f^*(\mathrm{Tot}(\mathcal{G}^\bullet)) \to \mathrm{Tot}(f_*f^*(\mathcal{G}^\bullet))$ is invertible. It suffices to show that it induces isomorphisms on homotopy groups

$$\pi_i f_*f^*(\mathrm{Tot}(\mathcal{G}^\bullet)) \to \pi_i \mathrm{Tot}(f_*f^*(\mathcal{G}^\bullet)) \tag{1.1}$$

for $i \geqslant 0$.

The fact that f is faithfully flat has the following consequences. First, the functor f_*f^* restricts to an exact functor between discrete objects, and $\pi_i f_*f^*(\mathcal{F}) = f_*f^*(\pi_i\mathcal{F})$ for each $\mathcal{F} \in \mathrm{Qcoh}(\mathrm{S})_{\geqslant 0}$ and $i \geqslant 0$. Second, a discrete object $\mathcal{F} \in \mathrm{Qcoh}(\mathrm{S})$ is zero iff $f_*f^*(\mathcal{F})$ is zero.

To compute the homotopy groups appearing in (1.1) we make use of the the Bousfield–Kan spectral sequence, in the form of the following lemma [2, Cor. 1.2.4.12]:

Lemma 1.3. *Let $\mathrm{E}^\bullet$ be a cosimplicial spectrum. Suppose that for each $i \geqslant 0$, the associated (unnormalized) cochain complex*

$$\pi_i(\mathrm{E}^0) \xrightarrow{\vartheta_i} \pi_i(\mathrm{E}^1) \to \pi_i(\mathrm{E}^2) \to \cdots$$

is an acyclic resolution of the kernel $\mathrm{K}_i = \mathrm{Ker}(\vartheta_i)$. Then for each $i \geqslant 0$, the map $\pi_i(\mathrm{Tot}(\mathrm{E}^\bullet)) \to \pi_i(\mathrm{E}_0)$ induces an isomorphism $\pi_i(\mathrm{Tot}(\mathrm{E}^\bullet)) \xrightarrow{\sim} \mathrm{K}_i$.

To apply this, let

$$\pi_i(\mathcal{G}^0) \xrightarrow{\vartheta_i} \pi_i(\mathcal{G}^1) \to \pi_i(\mathcal{G}^2) \to \cdots \tag{1.2}$$

denote the unnormalized cochain complex associated to $\pi_i(\mathcal{G}^\bullet)$, for each i. Since $f^*(\mathcal{G}^\bullet)$ is split, $\pi_i(f_*f^*\mathcal{G}^\bullet) = f_*f^*\pi_i(\mathcal{G}^\bullet)$ is a split cosimplicial object for each i. This implies that the image of (1.2) by the functor f_*f^* is split exact, so in particular the sequence

$$0 \to f_*f^*(\mathcal{K}_i) \to \pi_i(f_*f^*\mathcal{G}^0) \xrightarrow{\vartheta_i} \pi_i(f_*f^*\mathcal{G}^1) \to \pi_i(f_*f^*\mathcal{G}^2) \to \cdots$$

is exact, where $\mathcal{K}_i = \mathrm{Ker}(\vartheta_i)$. By the properties of f_*f^* discussed above, this implies that the sequence

$$0 \to \mathcal{K}_i \to \pi_i(\mathcal{G}^0) \xrightarrow{\vartheta_i} \pi_i(\mathcal{G}^1) \to \pi_i(\mathcal{G}^2) \to \cdots$$

is also exact. Applying the Lemma twice, we see that the map (1.1) is canonically identified with the identity of $f_*f^*\mathcal{K}_i$.

1.4. Restriction along the inclusion $\mathrm{DSch}^{\mathrm{aff}} \hookrightarrow \mathrm{DSch}$ preserves fpqc sheaves and defines, for any presentable ∞-category $\mathbf{V}$, a functor

$$\mathrm{Sh}_{\mathbf{V}}(\mathrm{DSch}) \to \mathrm{Sh}_{\mathbf{V}}(\mathrm{DSch}^{\mathrm{aff}}) \tag{1.3}$$

from $\mathbf{V}$-valued fpqc sheaves on DSch to $\mathbf{V}$-valued fpqc sheaves on $\mathrm{DSch}^{\mathrm{aff}}$.

As in classical algebraic geometry, the fact that any derived scheme admits a Zariski cover by affine schemes implies:

Proposition 1.5.

(i) *The canonical functor (1.3) is an equivalence.*

(ii) *Let $\mathcal{F}$ be an fpqc sheaf on* $\mathrm{DSch}^{\mathrm{aff}}$. *Then its right Kan extension to* DSch *is an fpqc sheaf.*

The first claim follows by applying an ∞-categorical version of the "comparison lemma", see e.g. [1, Lem. C.3]. The second follows from the fact that the inclusion $\mathrm{DSch}^{\mathrm{aff}} \hookrightarrow \mathrm{DSch}$ is topologically cocontinuous, i.e. every cover of $\mathrm{X} \in \mathrm{DSch}^{\mathrm{aff}}$ by derived schemes can be refined to a cover by affine derived schemes.

1.6. By Theorem 1.2 and Proposition 1.5 we deduce:

Corollary 1.7. *The presheaf of ∞-categories*

$$\mathrm{Qcoh} : (\mathrm{DSch})^{\mathrm{op}} \to \infty\text{-}\mathrm{Cat}$$

satisfies fpqc descent.

1.8. Let X be a derived scheme and let $(j_\alpha : \mathrm{U}_\alpha \hookrightarrow \mathrm{X})_\alpha$ be a Zariski-covering family. Then since the augmented cosimplicial diagram

$$\mathrm{Qcoh}(\mathrm{X}) \to \prod_\alpha \mathrm{Qcoh}(\mathrm{U}_\alpha) \rightrightarrows \prod_{\alpha,\beta} \mathrm{Qcoh}(\mathrm{U}_\alpha \cap \mathrm{U}_\beta) \rightrightarrows \cdots$$

is a limit diagram (Corollary 1.7), we immediately see:

Corollary 1.9. *The family of functors $(j_\alpha^*)_\alpha : \mathrm{Qcoh}(\mathrm{X}) \to \prod_\alpha \mathrm{Qcoh}(\mathrm{U}_\alpha)$ is conservative.*

2. The direct image functor. Let $f : \mathrm{X} \to \mathrm{Y}$ be a morphism of derived schemes. Recall that the functor f_*, right adjoint to f^*, was constructed by abstract nonsense. Under a mild assumption on f we can give a more concrete description of it.

2.1. Let $\mathcal{F}$ be a quasi-coherent sheaf on X, which is by definition a collection $(s^*\mathcal{F})_s$ for each affine derived scheme S and each morphism $s : \mathrm{S} \to \mathrm{X}$. We would like to define a quasi-coherent sheaf $f_*\mathcal{F}$ on Y such that for each affine derived scheme T and each morphism $t : \mathrm{T} \to \mathrm{Y}$, we have

$$t^*(f_*\mathcal{F}) := g_*s^*(\mathcal{F})$$

where we have a cartesian square

$$\begin{tikzcd} \mathrm{S} \arrow[r, "g"] \arrow[d, "s"] & \mathrm{T} \arrow[d, "t"] \\ \mathrm{X} \arrow[r, "f"] & \mathrm{Y}. \end{tikzcd}$$

In order for this to be a well-defined element of Qcoh(Y), we need the following. If T' is another affine derived scheme with morphisms $t' : \mathrm{T}' \to \mathrm{Y}$ and $h : \mathrm{T}' \to \mathrm{T}$ such that $t = t' \circ h$, then we want to have a canonical isomorphism $h^*g_*s^*(\mathcal{F}) = (g')_*(s')^*\mathcal{F}$, where $\mathrm{S}' = \mathrm{T}' \times_{\mathrm{Y}} \mathrm{X}$ and $g' : \mathrm{S}' \to \mathrm{T}'$ and $s' : \mathrm{S}' \to \mathrm{X}$. This would follow from a base change formula for the square

$$\begin{tikzcd} \mathrm{S}' \arrow[r, "g'"] \arrow[d, "h'"] & \mathrm{T}' \arrow[d, "h"] \\ \mathrm{S} \arrow[r, "g"] & \mathrm{T}. \end{tikzcd} \tag{2.1}$$

There is always a canonical natural transformation (exercise: write it down using the (co)units of the adjunctions)

$$h^*g_* \to (g')_*(h')^* \tag{2.2}$$

and we want to say that it is invertible.

In case the morphism f is *affine* (which means that for any affine scheme mapping into the target, the base change is also an affine scheme), then S and S' are affine. In this case the base change property is easy to check. Indeed, in the affine case it is clear that the functor g_* preserves colimits (it is left adjoint to the functor of "coextension of scalars") and that every quasi-coherent sheaf on S is built from $\mathcal{O}_{\mathrm{S}}$ using colimits. Therefore it suffices to check that (2.2) is an isomorphism for the quasi-coherent sheaf $\mathcal{O}_{\mathrm{S}}$, which is obvious.

More generally, one can prove:

Proposition 2.2. *Let $g : \mathrm{S} \to \mathrm{T}$ be a quasi-compact morphism of derived schemes. Then we have:*

(i) *For any cartesian square of derived schemes (2.1), the base change transformation (2.2) is invertible.*

(ii) *The functor g_* preserves colimits.*

By Zariski descent one reduces to the case where S is affine, and therefore the base change S' is quasi-compact. A cofinality argument allows a further reduction to the case where T and T' are quasi-compact. A quasi-compact scheme admits a finite affine Zariski cover, so one finally reduces to the affine case by an argument involving induction on the size of the cover. We omit the details but we will give a similar style of argument in the next lecture.

2.3. Let $j : \mathrm{U} \hookrightarrow \mathrm{X}$ be a quasi-compact open immersion of derived schemes. Then we have:

Corollary 2.4. *The functor $j_* : \mathrm{Qcoh}(\mathrm{U}) \to \mathrm{Qcoh}(\mathrm{X})$ is fully faithful.*

It suffices to show that the co-unit $j^* j_*(\mathcal{F}) \to \mathcal{F}$ is invertible for every $\mathcal{F} \in \mathrm{Qcoh}(\mathrm{U})$. This follows from the base change property (Proposition 2.2) applied to the square

$$\begin{tikzcd} \mathrm{U} \arrow[r, hook] \arrow[d] & \mathrm{U} \arrow[d] \\ \mathrm{U} \arrow[r, hook] & \mathrm{X} \end{tikzcd}$$

which is cartesian because j is a monomorphism.

3. Zariski excision. Let X be a derived scheme. By fpqc descent, the ∞-category Qcoh(X) can be computed as the totalization

$$\mathrm{Qcoh}(\mathrm{X}) = \mathrm{Tot}(\mathrm{Qcoh}(\check{\mathrm{C}}(\tilde{\mathrm{X}}/\mathrm{X})_\bullet)),$$

where $\tilde{\mathrm{X}} \to \mathrm{X}$ is an fpqc cover. In many situations it is extremely useful to be able to express Qcoh(X) as a *finite* limit. We now explain how to do this at least for the Zariski topology.

3.1. Let X be a derived scheme and suppose we have a cartesian square

$$\begin{tikzcd} \mathrm{U} \cap \mathrm{V} \arrow[r, hook, "j'_{\mathrm{U}}"] \arrow[d, hook, "j'_{\mathrm{V}}"] & \mathrm{V} \arrow[d, hook, "j_{\mathrm{V}}"] \\ \mathrm{U} \arrow[r, hook, "j_{\mathrm{U}}"] & \mathrm{X} \end{tikzcd}$$

where j_{U} and j_{V} are open immersions of derived schemes, and $\mathrm{X} = \mathrm{U} \cup \mathrm{V}$, i.e. the family $(j_{\mathrm{U}}, j_{\mathrm{V}})$ is Zariski-covering. As usual $\mathrm{U} \cap \mathrm{V}$ denotes the fibred product $\mathrm{U} \times_{\mathrm{X}} \mathrm{V}$.

Theorem 3.2. *The induced square of ∞-categories*

$$\begin{tikzcd} \mathrm{Qcoh}(\mathrm{X}) \arrow[r, "j_{\mathrm{U}}^*"] \arrow[d, "j_{\mathrm{V}}^*"] & \mathrm{Qcoh}(\mathrm{U}) \arrow[d, "(j'_{\mathrm{V}})^*"] \\ \mathrm{Qcoh}(\mathrm{V}) \arrow[r, "(j'_{\mathrm{U}})^*"] & \mathrm{Qcoh}(\mathrm{U} \cap \mathrm{V}) \end{tikzcd}$$

is cartesian.

It suffices to show that the canonical functor

$$u : \mathrm{Qcoh}(\mathrm{X}) \to \mathrm{Qcoh}(\mathrm{U}) \underset{\mathrm{Qcoh}(\mathrm{U}\cap\mathrm{V})}{\times} \mathrm{Qcoh}(\mathrm{V}) \tag{3.1}$$

is an equivalence. An object in the target consists of quasi-coherent sheaves $\mathcal{F}_{\mathrm{U}} \in \mathrm{Qcoh}(\mathrm{U})$, $\mathcal{F}_{\mathrm{V}} \in \mathrm{Qcoh}(\mathrm{V})$, and $\mathcal{F}_{\mathrm{U}\cap\mathrm{V}} \in \mathrm{Qcoh}(\mathrm{U} \cap \mathrm{V})$, together with isomorphisms $\alpha : (j'_{\mathrm{V}})^*\mathcal{F}_{\mathrm{U}} \to \mathcal{F}_{\mathrm{U}\cap\mathrm{V}}$ and $\beta : (j'_{\mathrm{U}})^*\mathcal{F}_{\mathrm{V}} \to \mathcal{F}_{\mathrm{U}\cap\mathrm{V}}$.

For formal reasons (the limit can be computed in the subcategory of presentable ∞-categories and left-adjoint functors), (3.1) has a right adjoint v which can be described explicitly as the functor

$$(\mathcal{F}_{\mathrm{U}}, \mathcal{F}_{\mathrm{V}}, \mathcal{F}_{\mathrm{U}\cap\mathrm{V}}, \alpha, \beta) \mapsto \mathrm{Cofib}((j_{\mathrm{U}})_*\mathcal{F}_{\mathrm{U}} \oplus (j_{\mathrm{V}})_*\mathcal{F}_{\mathrm{V}} \to (j_{\mathrm{U}\cap\mathrm{V}})_*\mathcal{F}_{\mathrm{U}\cap\mathrm{V}}). \tag{3.2}$$

It suffices to check that the unit and co-units of the adjunction are invertible, which is easy using Corollary 1.9.

Remark 3.3. Note that the argument only used the fact that Qcoh($-$) is a Zariski sheaf. More generally, it is true that any Zariski sheaf on DSch satisfies Zariski excision. In fact, one can show that the property of Zariski excision is *equivalent* to Zariski descent, at least when we restrict to quasi-coherent quasi-separated derived schemes. This is one of the nice features of the Zariski topology, which is shared by the Nisnevich but not by the étale or fpqc topologies.

References.

[1] Marc Hoyois, *A quadratic refinement of the Grothendieck–Lefschetz–Verdier trace formula.*

[2] Jacob Lurie, *Higher algebra*, version of 2016-05-16, available at `http://www.math.harvard.edu/~lurie/papers/HigherAlgebra.pdf`.

[3] Jacob Lurie, *Spectral algebraic geometry*, version of 2017-10-13, available at `http://www.math.harvard.edu/~lurie/papers/SAG-rootfile.pdf`.