Exercise sheet 2

1. Prove the Proposition from §1.3 in the lecture: every regular sequence in a ring A is Koszul-regular. (Hint: induction.)

2. Let k be a field, $\mathrm{A} = k[x]/\langle x^2 \rangle$. Show that k, viewed as an A-module, is not perfect.

3. Let $\phi : \mathrm{A} \to \mathrm{B}$ be a flat ring homomorphism.

(i) Show that if a f.g. A-module M is of Tor-amplitude $\leqslant n$, then so is the B-module $\mathrm{M} \otimes_{\mathrm{A}} \mathrm{B}$.

(ii) Suppose that ϕ is *faithfully* flat, i.e., that a sequence of A-modules $\mathrm{M}' \to \mathrm{M} \to \mathrm{M}''$ is exact iff $\mathrm{M}' \otimes_{\mathrm{A}} \mathrm{B} \to \mathrm{M} \otimes_{\mathrm{A}} \mathrm{B} \to \mathrm{M}'' \otimes_{\mathrm{A}} \mathrm{B}$ is exact. Show that a f.g. A-module M is of Tor-amplitude $\leqslant n$ if and only if $\mathrm{M} \otimes_{\mathrm{A}} \mathrm{B}$ is of Tor-amplitude $\leqslant n$.

4. Let A be a noetherian ring and M a finitely generated A-module. Show that M is of finite length iff $\mathrm{M}_{\mathfrak{p}} = 0$ for all non-maximal prime ideals $\mathfrak{p}$. (Use the Proposition in §1.3 of the lecture.)

The length *of an* A*-module* M *is the maximal length of a composition series (a filtration where the successive quotients are all simple, i.e., are nonzero and have no non-trivial, non-proper submodules). For example,* A *has length 1 iff* A *is a field. For a field, length coincides with dimension of vector spaces.*