CHOW GROUPS OF 0-CYCLES WITH MODULUS AND HIGHER DIMENSIONAL CLASS FIELD THEORY

Following Kerz-Saito, we introduce the Chow group of 0-cycles with modulus, relate it to the Wiesend idele class group, and show the existence of a canonical reciprocity map on the Chow group with modulus, with values in the abelianization of the fundamental group.

§ 1. CHOW GROUPS WITH MODULUS

We define the Chow group of 0-cycles with modulus.

1.1. Let X be a scheme over finite type over a perfect field k of characteristic $p > 0$. Let $j : \mathrm{X} \hookrightarrow \bar{\mathrm{X}}$ be an open immersion into a proper normal scheme $\bar{\mathrm{X}}$ over k, such that the complement $\bar{\mathrm{X}} - \mathrm{X}$ is the support of an effective Cartier divisor $\mathrm{C} \in \mathrm{Div}^+(\bar{\mathrm{X}})$. We will refer to such a datum $(\mathrm{X}, \bar{\mathrm{X}}, j)$ as a *compactified pair* over k.

1.2. Let $(\mathrm{X}, \bar{\mathrm{X}})$ be a compactified pair over k, and let $\mathrm{Z} \subset \bar{\mathrm{X}}$ be a one-dimensional integral closed subscheme not contained in $|\mathrm{C}|$. We will write $\psi_{\mathrm{Z}} : \mathrm{Z}^{\mathrm{N}} \to \mathrm{Z}$ for the normalization and

$$\mathrm{Z}_\infty := \psi_{\mathrm{N}}^{-1}(\mathrm{Z} \cap |\mathrm{C}|)$$

for the fibre. For a point $y \in \mathrm{Z}_\infty$, we will write $k(\mathrm{Z})_y$ for the henselization of $k(\mathrm{Z})$ at y. We will write

$$k(\mathrm{Z})_\infty^\times := \prod_{y \in \mathrm{Z}_\infty} k(\mathrm{Z})_y^\times .$$

In the sequel we will abuse notation by writing simply $\mathrm{Z} \subset \bar{\mathrm{X}}$ to mean that Z is as above.

1.3. Let $(\mathrm{X}, \bar{\mathrm{X}})$ be a compactified pair. Let $\mathrm{D} \in \mathrm{Div}^+(\bar{\mathrm{X}})$ be an effective Cartier divisor with $|\mathrm{D}| \subset \bar{\mathrm{X}} - \mathrm{X}$. For $\mathrm{Z} \subset \bar{\mathrm{X}}$, consider the canonical homomorphism

$$k(\mathrm{Z})^\times \longrightarrow \prod_{y \in \mathrm{Z}_\infty} k(\mathrm{Z})_y^\times / (1 + \mathscr{I}_{\mathrm{D}} \mathscr{O}_{\mathrm{Z}^{\mathrm{N}}, y})$$

where $\mathscr{I}_{\mathrm{D}} = \mathscr{O}_{\bar{\mathrm{X}}}(-\mathrm{D})$ denotes the ideal sheaf of D. We write $k(\mathrm{Z})_{\mathrm{D}}^\times$ for the kernel of this homomorphism.

1.4. ***Definition.*** — Let $(\mathrm{X}, \bar{\mathrm{X}})$ and D be as above. Consider the canonical homomorphism of abelian groups

$$\delta_{\mathrm{X}, \bar{\mathrm{X}}, \mathrm{D}} : \bigoplus_{\mathrm{Z} \subset \bar{\mathrm{X}}} k(\mathrm{Z})_{\mathrm{D}}^\times \longrightarrow \mathrm{Z}_0(\mathrm{X})$$

which sends an invertible rational function $f \in k(\mathrm{Z})^\times_{\mathrm{D}}$ to the 0-cycle

$$j_* \mathrm{div}(f|_{\mathrm{Z}\cap\mathrm{X}})$$

where $j : \mathrm{Z} \cap \mathrm{X} \hookrightarrow \mathrm{X}$ denotes the inclusion. The *Chow group of 0-cycles with modulus* D of the pair $(\mathrm{X}, \bar{\mathrm{X}})$ is defined as the cokernel of $\delta_{\mathrm{X},\bar{\mathrm{X}},\mathrm{D}}$, and is denoted

$$\mathrm{CH}_0(\mathrm{X}, \bar{\mathrm{X}}|\mathrm{D}).$$

1.5. ***Definition.*** — Let $(\mathrm{X}, \bar{\mathrm{X}})$ be a compactified pair over k. The *Chow group of 0-cycles with modulus* is defined as the projective limit

$$\mathrm{CH}_0^c(\mathrm{X}, \bar{\mathrm{X}}) := \varprojlim_{\mathrm{D}} \mathrm{CH}_0(\mathrm{X}, \bar{\mathrm{X}}|\mathrm{D})$$

over the codirected set of divisors $\mathrm{D} \in \mathrm{Div}^+(\bar{\mathrm{X}})$ with $|\mathrm{D}| \subset \bar{\mathrm{X}} - \mathrm{X}$. This projective limit is taken in the category of topological abelian groups, with each of the groups $\mathrm{CH}_0(\mathrm{X}, \bar{\mathrm{X}}|\mathrm{D})$ endowed with the discrete topology.

§ 2. THE WIESEND CLASS GROUP

We recall the definition of the Wiesend idele class group and identify a quotient of it with the Chow group with modulus.

2.1. ***Definition.*** — Let X be a scheme of finite type over k. Recall that the *Wiesend class group* $\mathrm{W}(\mathrm{X})$ is defined as the cokernel of the homomorphism of abelian groups

$$\bigoplus_{\mathrm{Z}\subset\mathrm{X}} k(\mathrm{Z})^\times \longrightarrow \mathrm{Z}_0(\mathrm{X}) \oplus \bigoplus_{\mathrm{Z}\subset\mathrm{X}} k(\mathrm{Z})^\times_\infty$$

induced by

$$\delta : \bigoplus_{\mathrm{Z}\subset\mathrm{X}} k(\mathrm{Z})^\times \longrightarrow \mathrm{Z}_0(\mathrm{X})$$

as defined in 1.4, and the canonical homomorphism

$$\bigoplus_{\mathrm{Z}\subset\mathrm{X}} k(\mathrm{Z})^\times \longrightarrow \bigoplus_{\mathrm{Z}\subset\mathrm{X}} k(\mathrm{Z})^\times_\infty.$$

2.2. Let $(\mathrm{X}, \bar{\mathrm{X}})$ be a compactified pair over k and $\mathrm{D} \in \mathrm{Div}^+(\bar{\mathrm{X}})$ with $|\mathrm{D}| \subset |\mathrm{C}|$. For $\mathrm{Z} \subset \bar{\mathrm{X}}$, consider the subgroup of $k(\mathrm{Z})^\times_\infty$ generated by

$$1 + \mathscr{I}_{\mathrm{D}} \mathscr{O}^h_{\mathrm{Z}^{\mathrm{N}},\mathrm{Z}\cap|\mathrm{C}|},$$

where $\mathscr{O}^h_{\mathrm{Z^N},\mathrm{Z}\cap|\mathrm{C}|}$ denotes the product of the henselian local rings $\mathscr{O}^h_{\mathrm{Z},y}$ for $y \in \mathrm{Z}_\infty$. Let

$$\hat{\mathrm{F}}^{(\mathrm{D})}\mathrm{W}(\mathrm{X},\bar{\mathrm{X}}) \subset \mathrm{W}(\mathrm{X})$$

denote the image of the direct sums of these subgroups by the canonical homomorphism

$$\bigoplus_{\mathrm{Z}\subset\mathrm{X}} k(\mathrm{Z})^\times_\infty \longrightarrow \mathrm{W}(\mathrm{X}).$$

2.3. ***Lemma.*** — *Let* $(\mathrm{X},\bar{\mathrm{X}})$ *be a compactified pair over* k *and* $\mathrm{D} \in \mathrm{Div}^+(\bar{\mathrm{X}})$ *with* $|\mathrm{D}| \subset |\mathrm{C}|$. *There is a canonical isomorphism*

$$\mathrm{CH}_0(\mathrm{X},\bar{\mathrm{X}}|\mathrm{D}) \xrightarrow{\sim} \mathrm{W}(\mathrm{X})/\hat{\mathrm{F}}^{(\mathrm{D})}\mathrm{W}(\mathrm{X},\bar{\mathrm{X}}).$$

Proof. — Consider the diagram of abelian groups with exact rows

$$\begin{tikzcd}
\bigoplus_{\mathrm{Z}\subset\bar{\mathrm{X}}} k(\mathrm{Z})^\times_\mathrm{D} \arrow[r] \arrow[d, hook] & \mathrm{Z}_0(\mathrm{X}) \arrow[r] \arrow[d, hook] & \mathrm{CH}_0(\mathrm{X},\bar{\mathrm{X}}|\mathrm{D}) \arrow[r] \arrow[d, dashed] & 0 \\
\bigoplus_{\mathrm{Z}\subset\bar{\mathrm{X}}} k(\mathrm{Z})^\times \arrow[r] & \mathrm{Z}_0(\mathrm{X}) \oplus \bigoplus_{\mathrm{Z}\subset\bar{\mathrm{X}}} k(\mathrm{Z})^\times_\infty \arrow[r] & \mathrm{W}(\mathrm{X}) \arrow[r] \arrow[d, two heads] & 0 \\
 & & \mathrm{W}(\mathrm{X})/\hat{\mathrm{F}}^{(\mathrm{D})}\mathrm{W}(\mathrm{X},\bar{\mathrm{X}}) &
\end{tikzcd}$$

The composite of the two vertical arrows on the right is a canonical homomorphism

$$\mathrm{CH}_0(\mathrm{X},\bar{\mathrm{X}}|\mathrm{D}) \longrightarrow \mathrm{W}(\mathrm{X})/\hat{\mathrm{F}}^{(\mathrm{D})}\mathrm{W}(\mathrm{X},\bar{\mathrm{X}}).$$

Injectivity follows from the definitions. Surjectivity follows from the weak approximation theorem.

2.4. ***Lemma.*** — *Let* $(\mathrm{X},\bar{\mathrm{X}})$ *be a compactified pair and* $f : \bar{\mathrm{X}}' \to \bar{\mathrm{X}}$ *a morphism with* $f(\bar{\mathrm{X}}') \cap \mathrm{X} \neq \emptyset$. *Let* $(\mathrm{X}',\bar{\mathrm{X}}')$ *be the compactified pair obtained as the inverse image of* $(\mathrm{X},\bar{\mathrm{X}})$ *along* f, *so that* $\mathrm{X}' = \mathrm{X} \times_{\bar{\mathrm{X}}} \bar{\mathrm{X}}'$. *For each* $\mathrm{D} \in \mathrm{Div}^+(\mathrm{X})$ *with* $|\mathrm{D}| \subset \bar{\mathrm{X}} - \mathrm{X}$, *the direct image homomorphism*

$$f^{\mathrm{W}}_* : \mathrm{W}(\mathrm{X}') \to \mathrm{W}(\mathrm{X})$$

induces a homomorphism

$$f^{\mathrm{D}}_* : \mathrm{CH}_0(\mathrm{X}',\bar{\mathrm{X}}'|f^*\mathrm{D}) \longrightarrow \mathrm{CH}_0(\mathrm{X},\bar{\mathrm{X}}|\mathrm{D}).$$

Proof. — One checks that

$$f^{\mathrm{W}}(\hat{\mathrm{F}}^{(f^*\mathrm{D})}\mathrm{W}(\mathrm{X}',\bar{\mathrm{X}}')) \subset \hat{\mathrm{F}}^{(\mathrm{D})}\mathrm{W}(\mathrm{X},\bar{\mathrm{X}}).$$

Indeed the modulus condition on these subgroups can be checked for all curves $\mathrm{Z} \subset \bar{\mathrm{X}}$ and for all points $y \in k(\mathrm{Z})_\infty$, so this follows from the classical theory of local fields.

2.5. ***Lemma.*** — *With the notation of the previous lemma, there is a canonical morphism of topological abelian groups*

$$f_* : \mathrm{CH}_0^c(\mathrm{X}',\bar{\mathrm{X}}') \to \mathrm{CH}_0^c(\mathrm{X},\bar{\mathrm{X}}).$$

Proof. — The direct image homomorphisms f_*^{D} of 2.4 induce a morphism of topological abelian groups

$$\varprojlim_{\mathrm{D}} \mathrm{CH}_0(\mathrm{X}',\bar{\mathrm{X}}'|f^*\mathrm{D}) \longrightarrow \varprojlim_{\mathrm{D}} \mathrm{CH}_0(\mathrm{X},\bar{\mathrm{X}}|\mathrm{D})$$

where the projective limit is taken over divisors $\mathrm{D} \in \mathrm{Div}^+(\bar{\mathrm{X}})$ with $|\mathrm{D}| \subset \bar{\mathrm{X}} - \mathrm{X}$. The right hand side is $\mathrm{CH}_0^c(\mathrm{X},\bar{\mathrm{X}})$ by definition, while the left hand side is canonically identified with $\mathrm{CH}_0^c(\mathrm{X}',\bar{\mathrm{X}}')$. Indeed it is sufficient to show that the codirected set of divisors $f^*\mathrm{D}$, with $\mathrm{D} \in \mathrm{Div}^+(\bar{\mathrm{X}})$ and $|\mathrm{D}| \subset \bar{\mathrm{X}} - \mathrm{X}$, is co-initial in the codirected set of divisors $\mathrm{D}' \in \mathrm{Div}^+(\bar{\mathrm{X}}')$ with $|\mathrm{D}'| \subset \bar{\mathrm{X}}' - \mathrm{X}'$. This is easy to check.

2.6. ***Proposition.*** — *The topological abelian group* $\mathrm{CH}_0^c(\mathrm{X},\bar{\mathrm{X}})$ *is independent of* $\bar{\mathrm{X}}$. *In particular it is covariantly functorial in* X.

Proof. —

Let $(\mathrm{X},\bar{\mathrm{X}}')$ be a second compactified pair. Suppose that there is a morphism $f : \bar{\mathrm{X}}' \to \bar{\mathrm{X}}$ which fixes X in the sense that the square

$$\begin{tikzcd} \mathrm{X} \arrow[r, hook] \arrow[d] & \bar{\mathrm{X}}' \arrow[d, "f"] \\ \mathrm{X} \arrow[r, hook] & \bar{\mathrm{X}} \end{tikzcd}$$

is cartesian. It is sufficient to show that the direct image morphism f_* of 2.5 is an isomorphism. For each D we have a commutative diagram of abelian groups with exact rows

$$\begin{tikzcd} \bigoplus_{\mathrm{Z}'\subset\bar{\mathrm{X}}'} k(\mathrm{Z}')^\times_{f^*\mathrm{D}} \arrow[r, "\varphi'"] \arrow[d] & \mathrm{Z}_0(\mathrm{X}) \arrow[r] \arrow[d] & \mathrm{CH}_0(\mathrm{X},\bar{\mathrm{X}}'|f^*\mathrm{D}) \arrow[r] \arrow[d, "f_*"] & 0 \\ \bigoplus_{\mathrm{Z}\subset\bar{\mathrm{X}}} k(\mathrm{Z})^\times_{\mathrm{D}} \arrow[r, "\varphi"] & \mathrm{Z}_0(\mathrm{X}) \arrow[r] & \mathrm{CH}_0(\mathrm{X},\bar{\mathrm{X}}|\mathrm{D}) \arrow[r] & 0. \end{tikzcd}$$

By the assumption on f, one sees that φ and φ' have the same image in $\mathrm{Z}_0(\mathrm{X})$. Indeed by definition, φ (resp. φ') depends only on the restriction of a rational function on $\mathrm{Z} \subset \bar{\mathrm{X}}$ (resp.

$\mathrm{Z} \subset \bar{\mathrm{X}}'$) to $\mathrm{Z} \cap \mathrm{X}$. It follows that the right hand morphism is invertible.

In general, we may consider the third compactified pair $(\mathrm{X}, \bar{\mathrm{Y}})$, where $\bar{\mathrm{Y}}$ is the normalization of the Zariski closure of X in $\bar{\mathrm{X}} \times \bar{\mathrm{X}}'$. Then there are canonical morphisms $\bar{\mathrm{Y}} \to \bar{\mathrm{X}}$ and $\bar{\mathrm{Y}} \to \bar{\mathrm{X}}'$, both fixing X. Hence the above argument shows that $(\mathrm{X}, \bar{\mathrm{X}})$ and $(\mathrm{X}, \bar{\mathrm{X}}')$ have the same Chow group with modulus as $(\mathrm{X}, \bar{\mathrm{Y}})$.

2.7. ***Remark.*** — In the sequel we will abuse notation and write

$$\mathrm{CH}_0^c(\mathrm{X}) := \mathrm{CH}_0^c(\mathrm{X}, \bar{\mathrm{X}})$$

for some choice of compactification.

§ 3. THE RECIPROCITY MAP

We define the reciprocity map on the Chow group with modulus with values in the abelianized fundamental group.

3.1. ***Proposition.*** — *Let* X *be a smooth variety over a finite field* k. *There exists a unique morphism of topological abelian groups*

$$\rho_{\mathrm{X}} : \mathrm{CH}_0^c(\mathrm{X}) \longrightarrow \pi_1^{\mathrm{ab}}(\mathrm{X})$$

with values in the abelianized fundamental group of X, *such that the diagram*

$$\begin{tikzcd} \mathrm{Z}_0(\mathrm{X}) \arrow[r, "\varphi"] \arrow[d] & \pi_1^{\mathrm{ab}}(\mathrm{X}) \\ \mathrm{CH}_0^c(\mathrm{X}) \arrow[ur, dashed, "\rho_{\mathrm{X}}"] & \end{tikzcd}$$

commutes, where φ *is induced by the Frobenius homomorphisms*

$$\mathrm{Z}_0(x) \longrightarrow \pi_1^{\mathrm{ab}}(\mathrm{X})$$

for closed points $x \in \mathrm{X}$.

Proof. —

Recall that we have the Wiesend reciprocity map

$$r_{\mathrm{X}} : \mathrm{W}(\mathrm{X}) \longrightarrow \pi_1^{\mathrm{ab}}(\mathrm{X}).$$

It is sufficient to show that it factors through

$$\begin{tikzcd}
\mathrm{W}(\mathrm{X}) \arrow[r, "r_{\mathrm{X}}"] \arrow[d] & \pi_1^{\mathrm{ab}}(\mathrm{X}) \\
\mathrm{CH}_0^c(\mathrm{X}) \arrow[ur, dashed, "\rho_{\mathrm{X}}"] &
\end{tikzcd}$$

By Pontryagin duality, it is equivalent to show that $r_{\mathrm{X}}^{\vee}$ factors through

$$\begin{tikzcd}
\mathrm{H}^1(\mathrm{X}) \arrow[r, dashed] \arrow[dr, "r_{\mathrm{X}}^{\vee}"] & \mathrm{CH}_0^c(\mathrm{X})^{\vee} \arrow[d] \\
& \mathrm{W}(\mathrm{X})^{\vee}
\end{tikzcd}$$

where we write $(-)^{\vee}$ for the Pontryagin duality functor. Indeed recall that there is a canonical identification $\pi_1^{\mathrm{ab}}(\mathrm{X})^{\vee} = \mathrm{H}^1(\mathrm{X}) := \mathrm{H}^1_{\mathrm{et}}(\mathrm{X}, \mathbf{Q}/b\mathrm{Z})$. Now it is sufficient to show that for all characters $\chi \in \mathrm{H}^1(\mathrm{X})$, the morphism $r_{\mathrm{X}}^{\vee}(\chi) : \mathrm{W}(\mathrm{X}) \to \mathbf{Q}/\mathbf{Z}$ factors through $\mathrm{CH}_0^c(\mathrm{X})$.

$$\begin{tikzcd}
\mathrm{W}(\mathrm{X}) \arrow[r, "r_{\mathrm{X}}^{\vee}(\chi)"] \arrow[d] & \mathbf{Q}/\mathbf{Z} \\
\mathrm{CH}_0^c(\mathrm{X}) \arrow[ur, dashed] &
\end{tikzcd}$$

For this it is sufficient to show that for all χ, and for some compactification $\mathrm{X} \hookrightarrow \bar{\mathrm{X}}$, there exists some effective Cartier divisor $\mathrm{D} \in \mathrm{Div}^+(\bar{\mathrm{X}})$ with support contained in $\bar{\mathrm{X}} - \mathrm{X}$, such that there is a factorization

$$\begin{tikzcd}
\mathrm{W}(\mathrm{X}) \arrow[r, "r_{\mathrm{X}}^{\vee}(\chi)"] \arrow[d] & \mathbf{Q}/\mathbf{Z} \\
\mathrm{CH}_0(\mathrm{X}, \bar{\mathrm{X}}|\mathrm{D}) \arrow[ur, dashed] &
\end{tikzcd}$$

commutes. Indeed by definition the projection $\mathrm{W}(\mathrm{X}) \twoheadrightarrow \mathrm{CH}_0(\mathrm{X}, \bar{\mathrm{X}}|\mathrm{D})$ factors through $\mathrm{CH}_0^c(\mathrm{X})$.

Now recall that by (kerz2013chow, Proposition 2.9), for any $\chi \in \mathrm{H}^1(\mathrm{X})$ there exists some such D with $\chi \in \mathrm{fil}_{\mathrm{D}}\mathrm{H}^1(\mathrm{X})$. It is sufficient to show that $r_{\mathrm{X}}^{\vee}(\chi)$ sends $\hat{\mathrm{F}}^{(\mathrm{D})}\mathrm{W}(\mathrm{X}, \bar{\mathrm{X}})$ to zero. That is, for all $\mathrm{Z} \subset \bar{\mathrm{X}}$, $f \in k(\mathrm{Z})_{\infty}^{\times}$ with f satisfying the modulus condition with respect to D, we want $r_{\mathrm{X}}^{\vee}(\chi)(f) = 0$. Note that f satisfying the modulus condition with respect to D is equivalent to $f_y \in \mathrm{U}^{(m)}_{k(\mathrm{Z})_y} \subset k(\mathrm{Z})_y^{\times}$ for all $y \in \mathrm{Z}_{\infty}$ and all $\mathrm{Z} \subset \bar{\mathrm{X}}$ (using the notation of (serre1962corps, chap. XV)).

Recall that by classical class field theory (serre1962corps, chap. XV, § 2) we have a commutative square

$$\begin{tikzcd} k(\mathrm{Z})_y^\times \arrow[r] \arrow[d, two heads] & \mathrm{Gal}^{\mathrm{ab}}(k(\mathrm{Z})_y) \arrow[d, two heads] \\ k(\mathrm{Z})_y^\times/\mathrm{U}_{k(\mathrm{Z})_y}^{(m)} \arrow[r] & \mathrm{Gal}^{\mathrm{ab}}(k(\mathrm{Z})_y)/\mathrm{G}_y^{(m)} \end{tikzcd}$$

where $\mathrm{G}_y^{(m)}$ denotes the ramification filtration as in loc. cit. Dualizing we get a commutative square

$$\begin{tikzcd} \mathrm{fil}_{m_y}\mathrm{H}^1(k(\mathrm{Z})_y) \arrow[r] \arrow[d, hook] & (k(\mathrm{Z})_y^\times/\mathrm{U}_{k(\mathrm{Z})_y}^{(m)})^\vee \arrow[d, hook] \\ \mathrm{H}^1(k(\mathrm{Z})_y) \arrow[r] & (k(\mathrm{Z})_y^\times)^\vee \end{tikzcd}$$

Since $\chi \in \mathrm{fil}_{\mathrm{D}}\mathrm{H}^1(\mathrm{X})$, we have $\chi|_{k(\mathrm{Z})_y} \in \mathrm{fil}_{m_y}\mathrm{H}^1(k(\mathrm{Z})_y)$ for all y and Z by definition. Hence $r_{\mathrm{X}}^\vee(\chi|_{k(\mathrm{Z})_y}) : k(\mathrm{Z})_y^\times \to \mathbf{Q}/\mathbf{Z}$ factors through $k(\mathrm{Z})_y^\times/\mathrm{U}_{k(\mathrm{Z})_y}^{(m)}$. The claim follows.

3.2. *Theorem.* — *Let* X *be a smooth variety over a finite field* k *with* $\mathrm{char}(k) \neq 2$. *Let* $\psi_{\mathrm{X}} : \mathrm{X} \to \mathrm{Spec}(k)$ *denote the structural morphism and let* $\mathrm{CH}_0^c(\mathrm{X})^0 \subset \mathrm{CH}_0^c(\mathrm{X})$ *denote the kernel of* $\psi_{\mathrm{X},*}$. *Similarly let* $\pi_1^{\mathrm{ab}}(\mathrm{X})^0 \subset \pi_1^{\mathrm{ab}}(\mathrm{X})$ *denote the kernel of* $\psi_{\mathrm{X},*} : \pi_1^{\mathrm{ab}}(\mathrm{X}) \to \pi_1^{\mathrm{ab}}(\mathrm{Spec}(k))$. *The induced morphism*

$$\rho_{\mathrm{X}}^0 : \mathrm{CH}_0^c(\mathrm{X})^0 \longrightarrow \pi_1^{\mathrm{ab}}(\mathrm{X})^0$$

is an isomorphism of topological abelian groups.

REFERENCES

kerz2013chow Kerz, Moritz and Saito, Shuji. *Chow group of 0-cycles with modulus and higher dimensional class field theory.*

serre1962corps Serre, Jean-Pierre. *Corps locaux.*