CHOW GROUPS OF 0-CYCLES WITH MODULUS AND HIGHER DIMENSIONAL CLASS FIELD THEORY

Following Kerz-Saito, we introduce the Chow group of 0-cycles with modulus, relate it to the Wiesend idele class group, and show the existence of a canonical reciprocity map on the Chow group with modulus, with values in the abelianization of the fundamental group.

§ 1. CHOW GROUPS WITH MODULUS

We define the Chow group of 0-cycles with modulus.

1.1. Let X be a scheme over finite type over a perfect field k of characteristic p > 0. Let $j : X \hookrightarrow \overline{X}$ be an open immersion into a proper normal scheme \overline{X} over k, such that the complement $\overline{X} - X$ is the support of an effective Cartier divisor $C \in \text{Div}^+(\overline{X})$. We will refer to such a datum (X, \overline{X}, j) as a *compactified pair* over k.

1.2. Let (X, \overline{X}) be a compactified pair over k, and let $Z \subset \overline{X}$ be a one-dimensional integral closed subscheme not contained in |C|. We will write $\psi_Z : Z^N \to Z$ for the normalization and

$$\mathbf{Z}_{\infty} := \psi_{\mathbf{N}}^{-1}(\mathbf{Z} \cap |\mathbf{C}|)$$

for the fibre. For a point $y \in \mathbb{Z}_{\infty}$, we will write $k(\mathbb{Z})_y$ for the henselization of $k(\mathbb{Z})$ at y. We will write

$$k(\mathbf{Z})_{\infty}^{\times} := \prod_{y \in \mathbf{Z}_{\infty}} k(\mathbf{Z})_{y}^{\times}.$$

In the sequel we will abuse notation by writing simply $Z \subset \overline{X}$ to mean that Z is as above.

1.3. Let (X, \overline{X}) be a compactified pair. Let $D \in \text{Div}^+(\overline{X})$ be an effective Cartier divisor with $|D| \subset \overline{X} - X$. For $Z \subset \overline{X}$, consider the canonical homomorphism

$$k(\mathbf{Z})^{\times} \longrightarrow \prod_{y \in \mathbf{Z}_{\infty}} k(\mathbf{Z})_{y}^{\times} / (1 + \mathscr{I}_{\mathbf{D}} \mathscr{O}_{\mathbf{Z}^{\mathbf{N}}, y})$$

where $\mathscr{I}_{\mathrm{D}} = \mathscr{O}_{\mathrm{X}}(-\mathrm{D})$ denotes the ideal sheaf of D. We write $k(\mathrm{Z})_{\mathrm{D}}^{\times}$ for the kernel of this homomorphism.

1.4. Definition. — Let (X, \overline{X}) and D be as above. Consider the canonical homomorphism of abelian groups

$$\delta_{\mathbf{X},\bar{\mathbf{X}},\mathbf{D}}: \bigoplus_{\mathbf{Z}\subset\bar{\mathbf{X}}} k(\mathbf{Z})_{\mathbf{D}}^{\times} \longrightarrow \mathbf{Z}_{0}(\mathbf{X})$$

which sends an invertible rational function $f \in k(\mathbf{Z})_{\mathbf{D}}^{\times}$ to the 0-cycle

$$j_* \operatorname{div}(f|_{\mathbb{Z} \cap \mathbb{X}})$$

where $j : \mathbb{Z} \cap \mathbb{X} \hookrightarrow \mathbb{X}$ denotes the inclusion. The *Chow group of 0-cycles with modulus* D of the pair (X, \overline{X}) is defined as the cokernel of $\delta_{X, \overline{X}, D}$, and is denoted

$$CH_0(X, \overline{X}|D).$$

1.5. Definition. — Let (X, \overline{X}) be a compactified pair over k. The Chow group of 0-cycles with modulus is defined as the projective limit

$$\operatorname{CH}_0^c(X,\bar{X}) := \varprojlim_D \operatorname{CH}_0(X,\bar{X}|D)$$

over the codirected set of divisors $D \in \text{Div}^+(\bar{X})$ with $|D| \subset \bar{X} - X$. This projective limit is taken in the category of topological abelian groups, with each of the groups $CH_0(X, \bar{X}|D)$ endowed with the discrete topology.

§ 2. THE WIESEND CLASS GROUP

We recall the definition of the Wiesend idele class group and identify a quotient of it with the Chow group with modulus.

2.1. Definition. — Let X be a scheme of finite type over k. Recall that the Wiesend class group W(X) is defined as the cokernel of the homomorphism of abelian groups

$$\bigoplus_{\mathbf{Z}\subset\mathbf{X}} k(\mathbf{Z})^{\times} \longrightarrow \mathbf{Z}_0(\mathbf{X}) \oplus \bigoplus_{\mathbf{Z}\subset\mathbf{X}} k(\mathbf{Z})_{\infty}^{\times}$$

induced by

$$\delta: \bigoplus_{\mathbf{Z} \subset \mathbf{X}} k(\mathbf{Z})^{\times} \longrightarrow \mathbf{Z}_0(\mathbf{X})$$

as defined in 1.4, and the canonical homomorphism

$$\bigoplus_{\mathbf{Z}\subset\mathbf{X}} k(\mathbf{Z})^{\times} \longrightarrow \bigoplus_{\mathbf{Z}\subset\mathbf{X}} k(\mathbf{Z})_{\infty}^{\times}.$$

2.2. Let (X, \overline{X}) be a compactified pair over k and $D \in \text{Div}^+(\overline{X})$ with $|D| \subset |C|$. For $Z \subset \overline{X}$, consider the subgroup of $k(Z)_{\infty}^{\times}$ generated by

$$1 + \mathscr{I}_{\mathcal{D}} \mathscr{O}^h_{\mathcal{Z}^{\mathcal{N}}, \mathcal{Z} \cap |\mathcal{C}|},$$

where $\mathscr{O}^h_{Z^N,Z\cap|C|}$ denotes the product of the henselian local rings $\mathscr{O}^h_{Z,y}$ for $y \in Z_{\infty}$. Let

$$\hat{\mathbf{F}}^{(\mathrm{D})}\mathbf{W}(\mathbf{X},\bar{\mathbf{X}}) \subset \mathbf{W}(\mathbf{X})$$

denote the image of the direct sums of these subgroups by the canonical homomorphism

$$\bigoplus_{\mathbf{Z}\subset\mathbf{X}}k(\mathbf{Z})_{\infty}^{\times}\longrightarrow \mathbf{W}(\mathbf{X}).$$

2.3. Lemma. — Let (X, \overline{X}) be a compactified pair over k and $D \in Div^+(\overline{X})$ with $|D| \subset |C|$. There is a canonical isomorphism

$$\operatorname{CH}_0(X, \overline{X}|D) \xrightarrow{\sim} W(X)/\widehat{F}^{(D)}W(X, \overline{X}).$$

Proof. — Consider the diagram of abelian groups with exact rows

The composite of the two vertical arrows on the right is a canonical homomorphism

$$CH_0(X, \overline{X}|D) \longrightarrow W(X)/\hat{F}^{(D)}W(X, \overline{X}).$$

Injectivity follows from the definitions. Surjectivity follows from the weak approximation theorem.

2.4. Lemma. — Let (X, \overline{X}) be a compactified pair and $f : \overline{X}' \to \overline{X}$ a morphism with $f(\overline{X}') \cap X \neq \emptyset$. Let (X', \overline{X}') be the compactified pair obtained as the inverse image of (X, \overline{X}) along f, so that $X' = X \times_{\overline{X}} \overline{X}'$. For each $D \in \text{Div}^+(X)$ with $|D| \subset \overline{X} - X$, the direct image homomorphism

$$f^{\mathrm{W}}_* : \mathrm{W}(\mathrm{X}') \to \mathrm{W}(\mathrm{X})$$

 $induces\ a\ homomorphism$

$$f^{\mathrm{D}}_* : \mathrm{CH}_0(\mathrm{X}', \bar{\mathrm{X}}'|f^*\mathrm{D}) \longrightarrow \mathrm{CH}_0(\mathrm{X}, \bar{\mathrm{X}}|\mathrm{D}).$$

Proof. — One checks that

$$f^{\mathrm{W}}(\hat{\mathrm{F}}^{(f^*\mathrm{D})}\mathrm{W}(\mathrm{X}',\bar{\mathrm{X}}')) \subset \hat{\mathrm{F}}^{(\mathrm{D})}\mathrm{W}(\mathrm{X},\bar{\mathrm{X}}).$$

Indeed the modulus condition on these subgroups can be checked for all curves $Z \subset \overline{X}$ and for all points $y \in k(Z)_{\infty}$, so this follows from the classical theory of local fields.

2.5. Lemma. — With the notation of the previous lemma, there is a canonical morphism of topological abelian groups

$$f_* : \operatorname{CH}_0^c(\mathbf{X}', \overline{\mathbf{X}}') \to \operatorname{CH}_0^c(\mathbf{X}, \overline{\mathbf{X}}).$$

Proof. — The direct image homomorphisms $f_*^{\rm D}$ of 2.4 induce a morphism of topological abelian groups

$$\varprojlim_{\mathbf{D}} \mathrm{CH}_{0}(\mathbf{X}', \bar{\mathbf{X}}' | f^{*} \mathbf{D}) \longrightarrow \varprojlim_{\mathbf{D}} \mathrm{CH}_{0}(\mathbf{X}, \bar{\mathbf{X}} | \mathbf{D})$$

where the projective limit is taken over divisors $D \in \text{Div}^+(\bar{X})$ with $|D| \subset \bar{X} - X$. The right hand side is $\text{CH}_0^c(X, \bar{X})$ by definition, while the left hand side is canonically identified with $\text{CH}_0^c(X', \bar{X}')$. Indeed it is sufficient to show that the codirected set of divisors f^*D , with $D \in \text{Div}^+(\bar{X})$ and $|D| \subset \bar{X} - X$, is co-initial in the codirected set of divisors $D' \in \text{Div}^+(\bar{X}')$ with $|D'| \subset \bar{X}' - X'$. This is easy to check.

2.6. Proposition. — The topological abelian group $CH_0^c(X, \overline{X})$ is independent of \overline{X} . In particular it is covariantly functorial in X.

Proof. —

Let (X, \overline{X}') be a second compactified pair. Suppose that there is a morphism $f : \overline{X}' \to \overline{X}$ which fixes X in the sense that the square

$$\begin{array}{c} \mathbf{X} \longleftrightarrow \bar{\mathbf{X}}' \\ \downarrow \qquad \qquad \downarrow^{f} \\ \mathbf{X} \longleftrightarrow \bar{\mathbf{X}} \end{array}$$

is cartesian. It is sufficient to show that the direct image morphism f_* of 2.5 is an isomorphism. For each D we have a commutative diagram of abelian groups with exact rows

$$\begin{array}{ccc} \bigoplus_{\mathbf{Z}' \subset \bar{\mathbf{X}}'} k(\mathbf{Z}')_{f^*\mathbf{D}}^{\times} & \stackrel{\varphi'}{\longrightarrow} \mathbf{Z}_0(\mathbf{X}) & \longrightarrow \operatorname{CH}_0(\mathbf{X}, \bar{\mathbf{X}}'|f^*\mathbf{D}) & \longrightarrow \mathbf{0} \\ & & & \downarrow & & \downarrow f_* \\ & \bigoplus_{\mathbf{Z} \subset \bar{\mathbf{X}}} k(\mathbf{Z})_{\mathbf{D}}^{\times} & \stackrel{\varphi}{\longrightarrow} \mathbf{Z}_0(\mathbf{X}) & \longrightarrow \operatorname{CH}_0(\mathbf{X}, \bar{\mathbf{X}}|\mathbf{D}) & \longrightarrow \mathbf{0}. \end{array}$$

By the assumption on f, one sees that φ and φ' have the same image in $Z_0(X)$. Indeed by definition, φ (resp. φ') depends only on the restriction of a rational function on $Z \subset \overline{X}$ (resp.

 $Z \subset \overline{X}'$ to $Z \cap X$. It follows that the right hand morphism is invertible.

In general, we may consider the third compactified pair (X, \bar{Y}) , where \bar{Y} is the normalization of the Zariski closure of X in $\bar{X} \times \bar{X}'$. Then there are canonical morphisms $\bar{Y} \to \bar{X}$ and $\bar{Y} \to \bar{X}'$, both fixing X. Hence the above argument shows that (X, \bar{X}) and (X, \bar{X}') have the same Chow group with modulus as (X, \bar{Y}) .

2.7. Remark. — In the sequel we will abuse notation and write

$$\mathrm{CH}_0^c(\mathrm{X}) := \mathrm{CH}_0^c(\mathrm{X}, \bar{\mathrm{X}})$$

for some choice of compactification.

§ 3. THE RECIPROCITY MAP

We define the reciprocity map on the Chow group with modulus with values in the abelianized fundamental group.

3.1. *Proposition.* — Let X be a smooth variety over a finite field k. There exists a unique morphism of topological abelian groups

$$\rho_{\mathbf{X}} : \mathrm{CH}_{0}^{c}(\mathbf{X}) \longrightarrow \pi_{1}^{\mathrm{ab}}(\mathbf{X})$$

with values in the abelianized fundamental group of X, such that the diagram

$$\begin{array}{c} \mathbf{Z}_0(\mathbf{X}) \xrightarrow{\varphi} \pi_1^{ab}(\mathbf{X}) \\ \downarrow & \stackrel{\rho_{\mathbf{X}}}{\longleftarrow} \\ \mathbf{CH}_0^c(\mathbf{X}) \end{array}$$

commutes, where φ is induced by the Frobenius homomorphisms

$$Z_0(x) \longrightarrow \pi_1^{ab}(X)$$

for closed points $x \in X$.

Proof. —

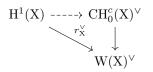
Recall that we have the Wiesend reciprocity map

$$r_{\mathbf{X}}: \mathbf{W}(\mathbf{X}) \longrightarrow \pi_1^{\mathrm{ab}}(\mathbf{X}).$$

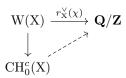
It is sufficient to show that it factors through

$$\begin{array}{c} W(X) \xrightarrow{r_X} \pi_1^{ab}(X) \\ \downarrow \xrightarrow{\rho_X} \\ CH_0^c(X) \end{array}$$

By Pontryagin duality, it is equivalent to show that r_X^{\vee} factors through



where we write $(-)^{\vee}$ for the Pontryagin duality functor. Indeed recall that there is a canonical identification $\pi_1^{ab}(X)^{\vee} = H^1(X) := H^1_{et}(X, \mathbf{Q}/bZ)$. Now it is sufficient to show that for all characters $\chi \in H^1(X)$, the morphism $r_X^{\vee}(\chi) : W(X) \to \mathbf{Q}/\mathbf{Z}$ factors through $\mathrm{CH}_0^c(X)$.



For this it is sufficient to show that for all χ , and for some compactification $X \hookrightarrow \overline{X}$, there exists some effective Cartier divisor $D \in \text{Div}^+(\overline{X})$ with support contained in $\overline{X} - X$, such that there is a factorization

$$\begin{array}{c} W(X) \xrightarrow{-r_X^{\vee}(\chi)} \mathbf{Q}/\mathbf{Z} \\ \downarrow \\ H_0(X, \bar{X}|D) \end{array}$$

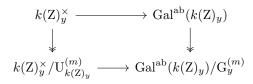
commutes. Indeed by definition the projection $W(X) \twoheadrightarrow CH_0(X, \overline{X}|D)$ factors through $CH_0^c(X)$.

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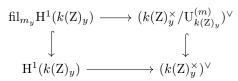
Now recall that by (kerz2013chow, Proposition 2.9), for any $\chi \in H^1(X)$ there exists some such D with $\chi \in fil_D H^1(X)$. It is sufficient to show that $r_X^{\vee}(\chi)$ sends $\hat{F}^{(D)}W(X, \bar{X})$ to zero. That is, for all $Z \subset \bar{X}$, $f \in k(Z)_{\infty}^{\times}$ with f satisfying the modulus condition with respect to D, we want $r_X^{\vee}(\chi)(f) = 0$. Note that f satisfying the modulus condition with respect to D is equivalent to $f_y \in U_{k(Z)_y}^{(m)} \subset k(Z)_y^{\times}$ for all $y \in Z_{\infty}$ and all $Z \subset \bar{X}$ (using the notation of (serre1962corps, chap. XV)).

Recall that by classical class field theory (serre1962corps, chap. XV, § 2) we have a

commutative square



where $G_y^{(m)}$ denotes the ramification filtration as in loc. cit. Dualizing we get a commutative square



Since $\chi \in \text{fil}_{D}\text{H}^{1}(X)$, we have $\chi|_{k(Z)_{y}} \in \text{fil}_{m_{y}}\text{H}^{1}(k(Z)_{y})$ for all y and Z by definition. Hence $r_{X}^{\vee}(\chi|_{k(Z)_{y}}): k(Z)_{y}^{\times} \to \mathbf{Q}/\mathbf{Z}$ factors through $k(Z)_{y}^{\times}/\text{U}_{k(Z)_{y}}^{(m)}$. The claim follows.

3.2. Theorem. — Let X be a smooth variety over a finite field k with $\operatorname{char}(k) \neq 2$. Let $\psi_X : X \to \operatorname{Spec}(k)$ denote the structural morphism and let $\operatorname{CH}_0^c(X)^0 \subset \operatorname{CH}_0^c(X)$ denote the kernel of $\psi_{X,*}$. Similarly let $\pi_1^{\operatorname{ab}}(X)^0 \subset \pi_1^{\operatorname{ab}}(X)$ denote the kernel of $\psi_{X,*} : \pi_1^{\operatorname{ab}}(X) \to \pi_1^{\operatorname{ab}}(\operatorname{Spec}(k))$. The induced morphism

$$\rho_{\mathbf{X}}^{0}: \mathrm{CH}_{0}^{c}(\mathbf{X})^{0} \longrightarrow \pi_{1}^{\mathrm{ab}}(\mathbf{X})^{0}$$

is an isomorphism of topological abelian groups.

REFERENCES

kerz2013chow Kerz, Moritz and Saito, Shuji. Chow group of 0-cycles with modulus and higher dimensional class field theory.

serre1962corps Serre, Jean-Pierre. Corps locaux.